64-16

SSD-TDR-63-367

Stresses in Thin Vessels Under Internal Pressure

10 JANUARY 1964

Prepared by NORMAN N. AU
Solid Mechanics Department

Prepared for COMMANDER SPACE SYSTEMS DIVISION

UNITED STATES AIR FORCE

Inglewood, California

ENGINEERING DIVISION • AEROSPACE CORPORATION
CONTRACT NO. AF 04(695)-269

SSD-TDR-63-367

Report No.
TDR-269(4304)-5
(formerly
ATN-63(3304)-1)

STRESSES IN THIN VESSELS UNDER INTERNAL PRESSURE

Prepared by

Norman N. Au
Solid Mechanics Department

AEROSPACE CORPORATION
El Segundo, California

Contract No. AF 04(695)-269

15 January 1964

Prepared for

COMMANDER SPACE SYSTEMS DIVISION
UNITED STATES AIR FORCE
Inglewood, California

SSD-TDR-63-367

Report No.
TDR-269(4304)-5

STRESSES IN THIN VESSELS UNDER INTERNAL PRESSURE

Prepared by:

Norman N. Au

Norman N. Au, Manager
Structures Section

Approved by:

J. Steinman

J. Steinman, Head
Solid Mechanics Department

This technical documentary report has been reviewed and is approved for publication and dissemination. The conclusions and findings contained herein do not necessarily represent an official Air Force position.

For Space Systems Division
Air Force Systems Command:

Raymond A. Maloney

Raymond A. Maloney
Major, USAF
SSYC

AEROSPACE CORPORATION
El Segundo, California

FOREWORD

The number of publications in the field of pressure vessel stress analysis has increased precipitously in recent years and the information contained therein, though pertinent, is, many times, lost in this sea of literature. In addition, a large proportion of the solutions, as presented, are not carried to the point where they can be used directly for determining stress distributions. The purpose of this report, then, is to make available a compact and adequate summary of the formulas and to present them in a readily usable fashion for the stress analysis of pressure vessels that are commonly encountered in missile design. That such a report as this derives largely from the work of others is self-evident, and it is the author's hope that due acknowledgement has been made of the immediate sources of all material here presented. However, while most publications deal with head closures joined to long circular cylindrical shell sections, this report places equal emphasis on designs in which the circular cylindrical shell section is short and is integrally joined to head closures of different shapes and thicknesses. The equations are organized such that interchangeability of head closures is a matter of algebraic manipulation. It is believed that the material contained in this report will be helpful to those involved in pressure vessel stress analysis and will be particularly useful for members of the Structures Section, Solid Mechanics Department, who are often asked to appraise pressure vessel design speedily and yet with sufficient accuracy as to insure a degree of confidence in their appraisal. The material presented herein can also be used to spot check stress levels obtained from computer solutions.

The author would like to take this opportunity to express his thanks to Mr. Walter Smotrys of the Structures Section for his assistance in checking the algebraic manipulations associated with the preparation of this report. He also wished to thank Miss M. J. McNeil for her computations of the frequently encountered parameters and the preparation of the associated graphs.

Finally, it should be said that, although every care has been taken to avoid errors, it would be over-sanguine to hope that none had escaped detection; for any suggestions that readers may make concerning needed correction, the author will be grateful.

SSD-TDR-63-367

Report No.
TDR-269(4304)-5

STRESSES IN THIN VESSELS UNDER INTERNAL PRESSURE*

ABSTRACT

Elastic stresses in thin shells of revolution under the action of internal pressure are presented. The formulas given are developed on the basis of Love's classical shell theory. The pressure vessel configurations under consideration consist of various commonly encountered head closure designs integrally joined to circular cylindrical shell sections which may be classified as long where the characteristic length of $\beta_c l \geq 4.0$ or short where $\beta_c l < 4.0$. In addition to the membrane stresses, the bending stresses resulting from forces and moments at the junctures of the heads and cylinders are also presented. The concept of edge influence numbers is used where convenient to express the discontinuity forces and moments at the junction. Many important parameters are expressed in graphical forms to facilitate analysis.

*This work completed March, 1963.

TABLE OF CONTENTS

TABLE OF CONTENTS (Cont'd)

TABLE OF CONTENTS (Cont'd)

TABLE OF CONTENTS (Cont'd)

1. INTRODUCTION

In this report, a pressure vessel will be defined as a container that must withstand an internal pressure, and is in the form of a surface of revolution with a wall thickness small compared to the radii of curvature of the wall ($h/a \leq 1/15$). It is well known that in every boost vehicle for space launching and ballistic missile systems, the pressure vessels comprise a large percentage of the total system structural weight. Consequently, their design merits considerable attention from the analyst. In flight, liquid systems will be pressurized either by the acceleration forces acting on the hydrostatic head of the propellant plus a pressure head for efficient pumping of the propellants to the engines or, in the case of solid propellants, by the pressure of the burning propellant gases. In this report, however, attention will be confined to the effects of loadings caused by internal pressure only.

Pressure vessels with very thin walls which offer no resistance to bending would be subjected only to direct stresses uniformly distributed through the thickness. In other words, the wall is acting as a membrane. The associated stresses are called "membrane stresses."

However, when the wall offers resistance to bending, bending stresses occur in addition to the membrane stresses. In pressure vessel design, bending stresses arise as a result of (a) change in curvature, (b) change in slope, and (c) change in wall thickness. Bending stresses resulting from these causes are called "discontinuity stresses." These stresses are obviously most severe at or near the discontinuity. They do not vary circumferentially because of the presumably axial symmetry of the structures and decay rapidly to negligible values in a distance $\ell \geq 4\beta$ measured from the point of discontinuity on the meridional arc.

From the above brief description, it is clear that in order to determine the complete state of stress in a pressure vessel, it is necessary to find the membrane and the discontinuity stresses. These separate effects may be linearly superimposed.

It is well known to those familiar with the calculation of bending stresses in shells that the process involved is both complex and time consuming. To a great extent, the complexity is associated with the determination of the discontinuity circumferential shear forces and bending moments existent at the juncture of head closures and cylinder or at other discontinuities mentioned earlier. Accordingly, a large portion of this report will be devoted to the calculation of these discontinuity forces and bending moments. The concept of edge influence numbers is used where convenient for this purpose. (By "edge influence number" is meant the displacement or rotation of the edge of the shell due to unit values of the edge bending loads or unit values of the pressure.) This concept is not new. Indeed, Müller-Breslau used it many years ago for beams, trusses, etc., and other authors have extended it to shells, such as the works of Galletly (Ref. 1); Watts and Lang (Ref. 2); Taylor and Wenk (Ref. 3); to mention a few. Having the edge influence numbers will greatly simplify the stress analyst's task in formulating the compatibility equations at the junctions of head closures and cylinders.

Appropriate expressions have been included in this report to permit the determination of stress distribution throughout the shell.

The following assumptions are employed throughout this report:

(a) The material under consideration is assumed to be a perfectly elastic, homogeneous, and isotropic solid.

(b) The load is assumed to be entirely due to internal pressure, so that support, dead weight, and similar loads are completely neglected in this report.

(c) All heads are considered to be complete (without holes) and to be free from any stress raisers other than the head-cylinder juncture itself.

(d) Each cylinder and head closure is assumed to have uniform thickness (although the thicknesses need not be the same).

(e) The middle surfaces of the cylinder and head at the juncture are assumed to be continuous.

Before we proceed to the main portion of the report, it is important to recognize that the formulas presented are derived on the basis of the elastic behavior of the shell materials, so that when the highly localized discontinuity stresses calculated by these expressions exceed the elastic limit of the material, local plastic deformation in the region of the junction of the cylinder and head will take place. This plastic deformation prevents stresses of the calculated intensities and, in areas where it is possible for plastic flow to occur, the maximum stress may be limited; the high peaks indicated by elastic analysis will generally be redistributed and increase the stress at nearby points.

One might therefore challenge the value of the elastic analysis. The following statements may be made in its defense:

(a) The high stresses indicated by elastic analysis are valuable since they point out potential trouble spots.

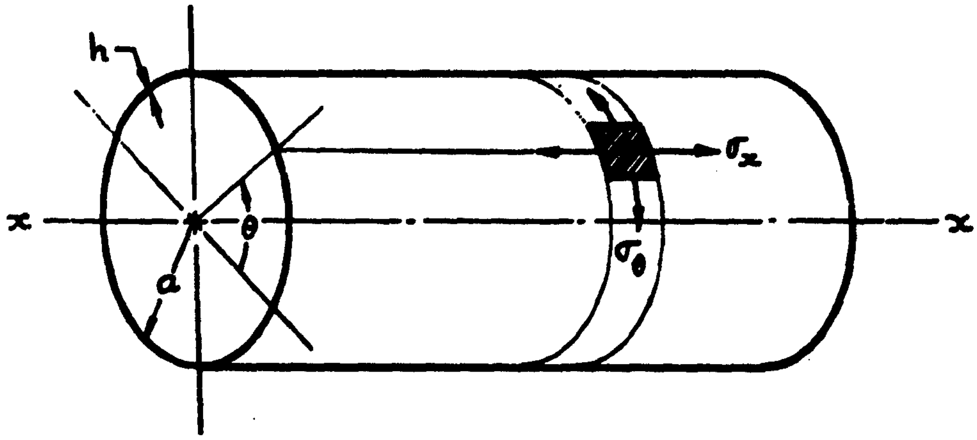
(b) The elastic analysis is needed to define the areas where plastic flow may occur.

(c) Although high local stresses in steel or aluminum vessels are frequently relieved by plastic flow, a good design cannot always rely on ductility as insurance against the bad effects of sharp corners and other stress raisers. For low temperature operations, some normally ductile materials become brittle and failure as a result of this brittleness can occur.

2. MEMBRANE THEORY SOLUTIONS

A tabulation of Membrane Theory solutions for pressure vessel configurations of practical interest under the action of internal pressure is offered in this section.

2.1 Thin Circular Cylindrical Shell Under Internal Pressure



$$\sigma_\theta = \frac{pa}{h}$$

$$\sigma_x = \frac{pa}{2h}$$

$$u_r = \frac{pa^2}{Eh}\left(1 - \frac{\nu}{2}\right)$$

where σ_θ = hoop membrane stress

σ_x = meridional membrane stress

u_r = radial displacement

p = uniform internal pressure

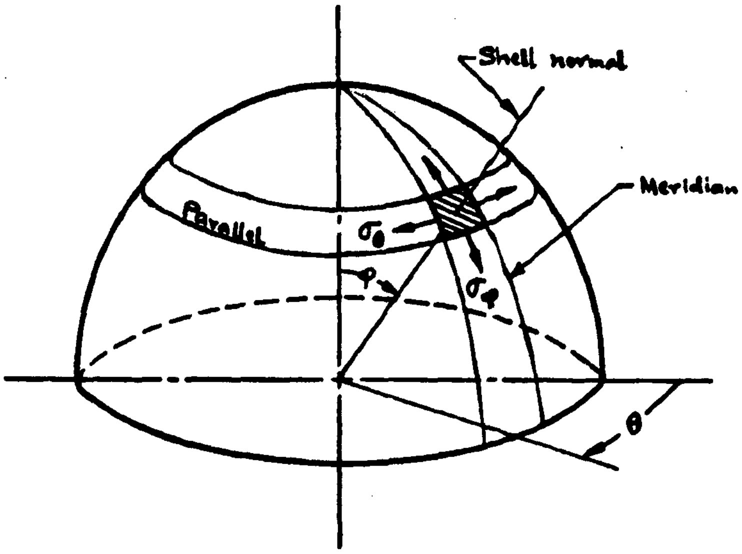
a = mean radius of curvature

h = shell thickness

E = modulus of elasticity

ν = Poisson's ratio

2.2 Thin Spherical Shell Under Internal Pressure



$$\sigma_\theta = \sigma_\phi = \frac{pa}{2h}$$

$$u_r = \frac{pa^2}{2Eh}(1 - \nu)$$

where σ_θ = hoop membrane stress

σ_ϕ = meridional membrane stress

u_r = radial displacement

p = uniform internal pressure

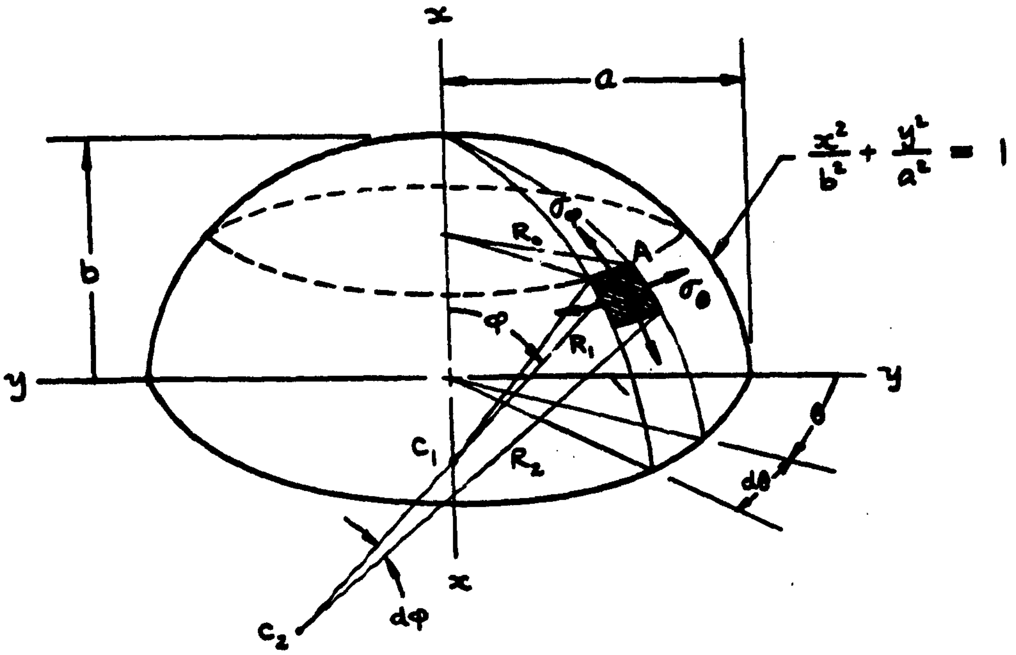
a = mean radius of the spherical shell

h = shell thickness

E = modulus of elasticity

ν = Poisson's ratio

2.3 Thin Ellipsoidal Shell Under Internal Pressure



$$R_1 = \frac{(a^4x^2 + b^4y^2)^{1/2}}{b^2}$$

$$R_2 = \frac{(a^4x^2 + b^4y^2)^{3/2}}{a^4b^4}$$

$$\sigma_\phi = \frac{pR_1}{2h}$$

$$= \frac{p(a^4x^2 + b^4y^2)^{1/2}}{2b^2h}$$

$$\sigma_\theta = \frac{pR_1}{h}\left(1 - \frac{R_1}{2R_2}\right)$$

$$= \frac{p(a^4x^2 + b^4y^2)^{1/2}}{2b^2h}\left(2 - \frac{a^4b^2}{a^4x^2 + b^4y^2}\right)$$

$$u_r = \frac{R_o}{E}(\sigma_\theta - \nu\sigma_\phi)$$

$$= \frac{p \sin\phi}{2b^4Eh}\left[(2 - \nu)(a^4x^2 + b^4y^2) - a^4b^2\right]$$

(a) <u>At the equator</u>: $y = a$, $x = 0$, $\phi = 90^\circ$.

$$\sigma_\phi = \frac{pa}{2h}$$

$$\sigma_\theta = \frac{pa}{h}\left(1 - \frac{a^2}{2b^2}\right)$$

$$u_r = \frac{pa^2}{2Eh}\left[(2 - \nu) - \frac{a^2}{b^2}\right]$$

(b) <u>At the top</u>: $y = 0$, $x = b$, $\phi = 0^\circ$.

$$\sigma_\phi = \sigma_\theta = \frac{pa^2}{2bh}$$

$$u_r = 0$$

where σ_θ = hoop membrane stress

σ_ϕ = meridional membrane stress

u_r = radial displacement

p = uniform internal pressure

a = semi-major axis of the ellipsoidal head, measured to the middle surface of the shell

b = semi-minor axis of the ellipsoidal head, measured to the middle surface of the shell

h = shell thickness

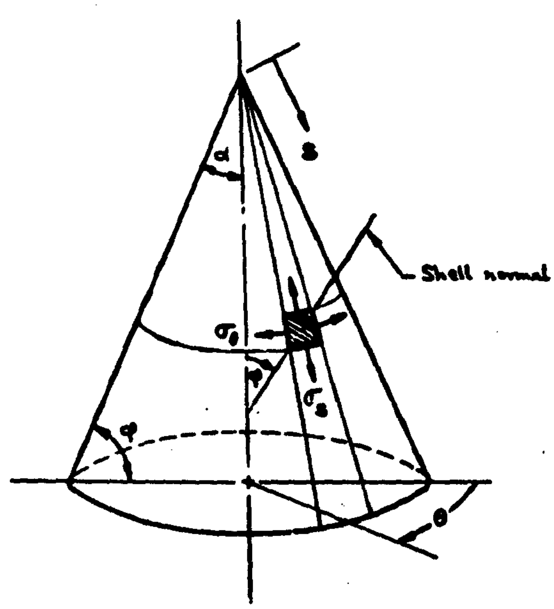
E = modulus of elasticity

ν = Poisson's ratio

$\left.\begin{matrix} R_1 \\ R_2 \end{matrix}\right\}$ Principal radii of curvature of the ellipsoid

$\left.\begin{matrix} x \\ y \end{matrix}\right\}$ Cartesian coordinate system

2.4 Thin Conical Shell Under Internal Pressure



$$\sigma_\theta = \frac{ps}{h} \tan \alpha$$

$$\sigma_s = \frac{ps}{2h} \tan \alpha$$

$$u_r = \frac{ps^2 \sin \alpha \tan \alpha}{2Eh} (2 - \nu)$$

where σ_θ = hoop membrane stress

σ_s = meridional membrane stress

u_r = radial displacement

p = uniform internal pressure

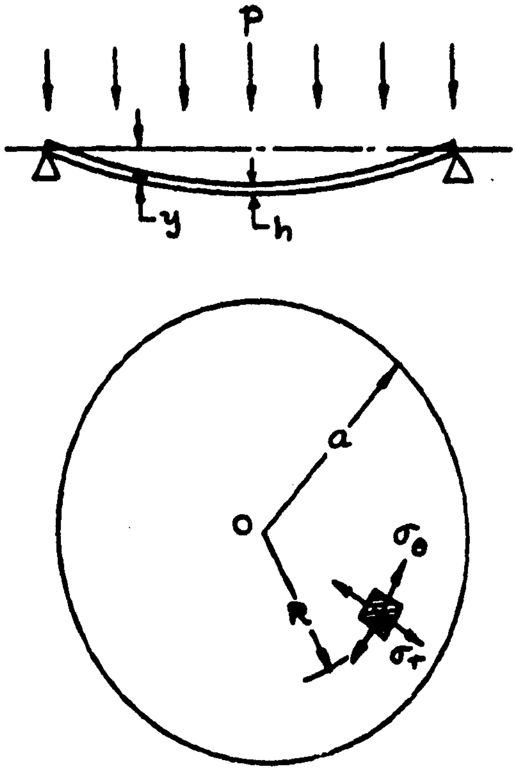
h = shell thickness

s = distance of a point of the middle surface from the vertex measured along a generator

E = modulus of elasticity

ν = Poisson's ratio

2.5 Thin Elastic Simply Supported Circular Plate Under Uniform Load Over Entire Surface



At R:

$$\sigma_r = \frac{3pa^2}{8h^2}(3+\nu)\left(1 - \frac{R^2}{a^2}\right)$$

$$\sigma_\theta = \frac{3pa^2}{8h^2}\left[(3+\nu) - (1+3\nu)\frac{R^2}{a^2}\right]$$

$$y = \frac{3pa^2(1-\nu)}{8Eh^3}\left[\frac{a^2}{2}(5+\nu) + \frac{R^4}{2a^2}(1+\nu) - R^2(3+\nu)\right]$$

<u>At center</u>: σ_r, σ_θ, and y attain their respective maximum values, and are given by:

$$\sigma_{r_{max}} = \sigma_{\theta_{max}} = \frac{3pa^2 (3 + \nu)}{8h^2}$$

$$y_{max} = \frac{3pa^4 (1 - \nu)(5 + \nu)}{16 Eh^3}$$

where σ_r = unit stress at surface of plate in the radial direction and is positive with tension at the lower surface and equal compression at the upper surface

σ_θ = unit stress at surface of plate in the tangential direction and is positive with tension at the lower surface and equal compression at the upper surface

y = vertical deflection of plate from original position and is positive for downward deflection

p = uniform pressure over the surface of the plate

h = plate thickness

E = modulus of elasticity

ν = Poisson's ratio

a = outside radius of plate

R = distance to any given point on surface of plate

3. BENDING THEORY SOLUTIONS

A tabulation of Bending Theory Solutions for pressure vessel configurations of practical interest under the action of various edge loads is offered in this section.

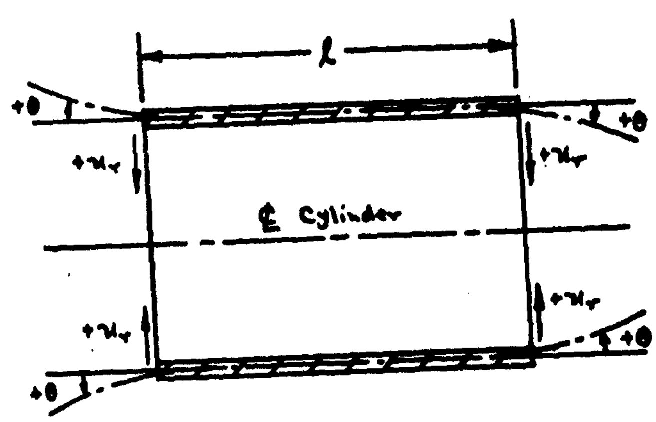
3.1 Circular Cylindrical Shells

The following signs convention for the rotation and deflection are used:

Positive Deflection - radially inward with respect to the center line of the cylinder

Positive Rotation - clockwise viewing the upper cut of cylinder

These signs are graphically represented below:



Signs Convention

The following nomenclature is used in this section:

σ_x = meridional stress

σ_θ = circumferential stress

u_r = radial displacement

θ = slope

$$\beta^4 = \frac{3(1-\nu^2)}{a^2h^2}$$

D = flexural rigidity of the shell

$$= \frac{Eh^3}{12(1-\nu^2)}$$

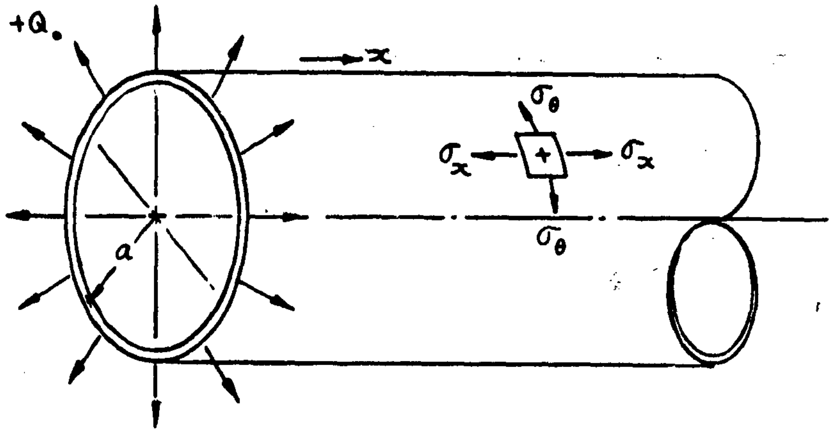
Where ± signs occur in the expressions for stresses, the upper sign refers to the inner surface of the cylinder and the lower sign to the outer cylinder.

The following notations are introduced for convenience:

Notation	Description	Plotted on Page
Ω_1	$e^{-\beta x} \sin \beta x$	63
Ω_2	$e^{-\beta x} \cos \beta x$	64
Ω_3	$e^{-\beta x}(\sin \beta x + \cos \beta x)$	65
Ω_4	$e^{-\beta x}(\cos \beta x - \sin \beta x)$	66
Ω_5	$\sin \beta x \sinh \beta x$	67
Ω_6	$\sin \beta x \cosh \beta x$	68
Ω_7	$\cos \beta x \sinh \beta x$	69
Ω_8	$\cos \beta x \cosh \beta x$	70
Ω_9	$\dfrac{\sin \frac{\beta l}{2} \sinh \frac{\beta l}{2}}{\sin \beta l + \sinh \beta l}$	71
Ω_{10}	$\dfrac{\sin \frac{\beta l}{2} \cosh \frac{\beta l}{2}}{\sin \beta l + \sinh \beta l}$	72
Ω_{11}	$\dfrac{\cos \frac{\beta l}{2} \sinh \frac{\beta l}{2}}{\sin \beta l + \sinh \beta l}$	73

Notation	Description	Plotted on Page
Ω_{12}	$\dfrac{\cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}}{\sin\beta l+\sinh\beta l}$	74
Ω_{13}	$\dfrac{\sinh\beta l-\sin\beta l}{\sin\beta l+\sinh\beta l}$	75
Ω_{14}	$\dfrac{\cosh\beta l+\cos\beta l}{\sin\beta l+\sinh\beta l}$	76
Ω_{15}	$\dfrac{\cosh\beta l-\cos\beta l}{\sin\beta l+\sinh\beta l}$	77
Ω_{16}	$\dfrac{\sinh\beta l\cosh\beta l-\sin\beta l\cos\beta l}{\sinh^2\beta l-\sin^2\beta l}$	78
Ω_{17}	$\dfrac{\sinh\beta l\cosh\beta l+\sin\beta l\cos\beta l}{\sinh^2\beta l-\sin^2\beta l}$	79
Ω_{18}	$\dfrac{\cos\beta l\sinh\beta l-\sin\beta l\cosh\beta l}{\sinh^2\beta l-\sin^2\beta l}$	80
Ω_{19}	$\dfrac{\cos\beta l\sinh\beta l+\sin\beta l\cosh\beta l}{\sinh^2\beta l-\sin^2\beta l}$	81
Ω_{20}	$\dfrac{\sinh^2\beta l+\sin^2\beta l}{\sinh^2\beta l-\sin^2\beta l}$	82
Ω_{21}	$\dfrac{\sin\beta l\sinh\beta l}{\sinh^2\beta l-\sin^2\beta l}$	83
Ω_{22}	$\dfrac{\sin^2\beta l}{\sinh^2\beta l-\sin^2\beta l}$	84
Ω_{23}	$\dfrac{\sinh^2\beta l}{\sinh^2\beta l-\sin^2\beta l}$	85

3.1.a Uniform Radial Shear Q_0 Lb Per Linear Inch of Circumference at End of Long Circular Cylindrical Shell



$$\sigma_x = \pm \frac{6Q_o}{\beta h^2}\Omega_1$$

$$\sigma_\theta = \frac{2Q_o}{h}(\beta a \Omega_2 \pm \frac{3\nu}{\beta h}\Omega_1)$$

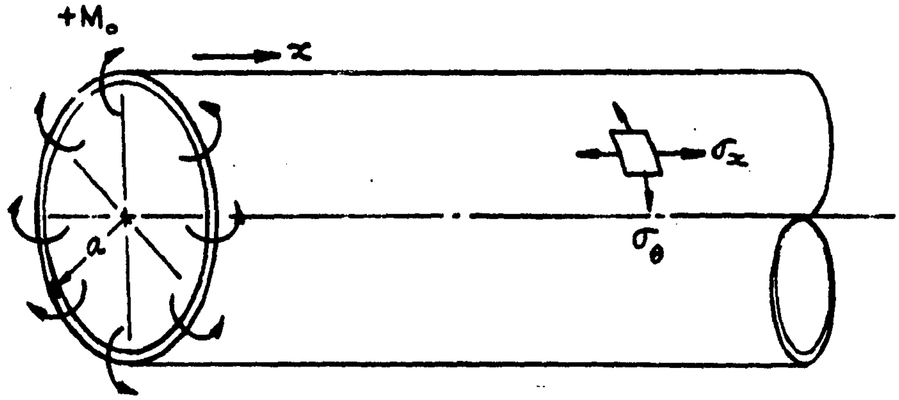
$$u_r = -\frac{Q_o}{2\beta^3 D}\Omega_2$$

$$\theta = \frac{Q_o}{2\beta^2 D}\Omega_3$$

$$u_{r(x=o)} = -\frac{Q_o}{2\beta^3 D}$$

$$\theta_{(x=o)} = \frac{Q_o}{2\beta^2 D}$$

3.1.b Uniform Radial Moment M_o In-Lb Per Linear Inch of Circumference at End of Long Circular Cylindrical Shell



$$\sigma_x = \pm \frac{GM_o}{h^2} \Omega_3$$

$$\sigma_\theta = \frac{2M_o}{h}\left(\pm \frac{3\nu}{h} \Omega_3 + \beta^2 a \Omega_4\right)$$

$$u_r = - \frac{M_o}{2\beta^2 D} \Omega_4$$

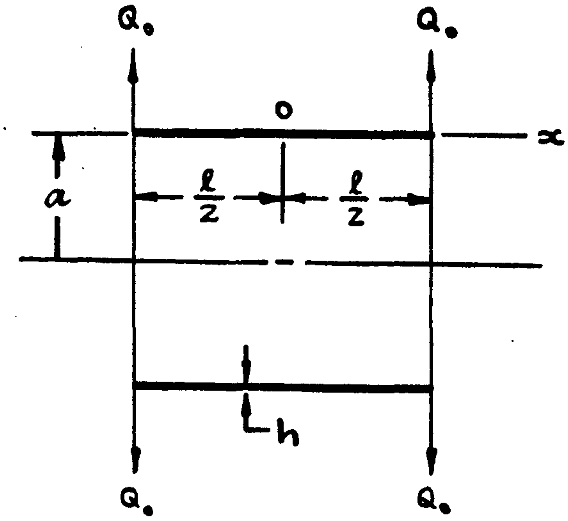
$$\theta = \frac{M_o}{\beta D} \Omega_2$$

$$u_{r(x=o)} = - \frac{M_o}{2\beta^2 D}$$

$$\theta_{(x=o)} = \frac{M_o}{\beta D}$$

3.1.c Short Circular Cylindrical Shell Bent by Forces Distributed Along The Edges

1) **Equal Edge Forces**



$$\sigma_x = \pm \frac{12Q_o}{\beta h^2}(\Omega_8\Omega_9 - \Omega_5\Omega_{12})$$

$$\sigma_\theta = \frac{4Q_o}{h}\left[\beta a(\Omega_5\Omega_9 + \Omega_8\Omega_{12}) \pm \frac{3\nu}{\beta h}(\Omega_8\Omega_9 - \Omega_5\Omega_{12})\right]$$

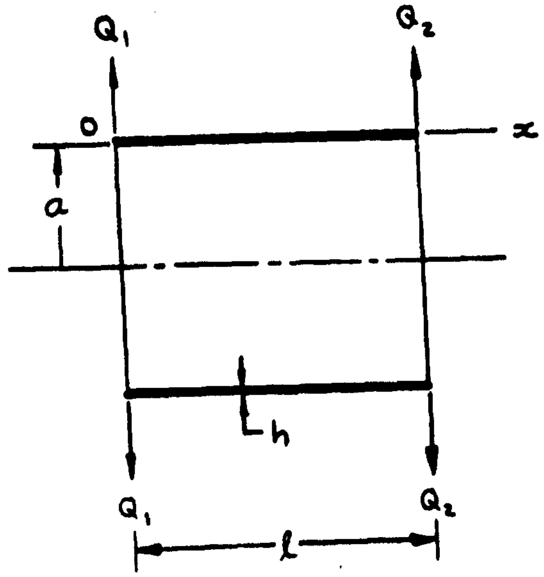
$$u_r = -\frac{Q_o}{\beta^3 D}(\Omega_5\Omega_9 + \Omega_8\Omega_{12})$$

$$\theta = -\frac{Q_o}{\beta^2 D}\left[\Omega_7(\Omega_9 + \Omega_{12}) + \Omega_6(\Omega_9 - \Omega_{12})\right]$$

$$u_{r(x=\pm l/2)} = -\frac{Q_o}{2\beta^3 D}\Omega_{14}$$

$$\theta_{(x=\pm l/2)} = \mp\frac{Q_o}{2\beta^2 D}\Omega_{13}$$

2) **Unequal Edge Forces**



$$\sigma_x = \pm\frac{6}{\beta h^2}\Big[(\Omega_{22}Q_2 - \Omega_{21}Q_1)\Omega_7 - (\Omega_{21}Q_2 - \Omega_{22}Q_1)\Omega_6 - (\Omega_{18}Q_2 + \Omega_{16}Q_1)\Omega_5\Big]$$

$$\sigma_\theta = \frac{2}{h}\Big\{\beta a\Big[\Omega_6(\Omega_{21}Q_2 - \Omega_{22}Q_1) + \Omega_7(\Omega_{21}Q_2 - \Omega_{23}Q_1) + \Omega_8(\Omega_{18}Q_2 + \Omega_{16}Q_1)\Big] \pm \frac{3\nu}{\beta h}\Big[\Omega_7(\Omega_{21}Q_2 - \Omega_{22}Q_1) - \Omega_6(\Omega_{21}Q_2 - \Omega_{23}Q_1) - \Omega_5(\Omega_{18}Q_2 - \Omega_{16}Q_1)\Big]\Big\}$$

$$u_r = -\frac{1}{2\beta^3 D}\Big[(\Omega_{21}Q_2 - \Omega_{22}Q_1)\Omega_6 + (\Omega_{21}Q_2 - \Omega_{23}Q_1)\Omega_7 + (\Omega_{18}Q_2 + \Omega_{16}Q_1)\Omega_8\Big]$$

$$\theta = -\frac{1}{2\beta^2 D}\Big[(\Omega_{21}Q_2 - \Omega_{22}Q_1)(\Omega_5 + \Omega_8) + (\Omega_{21}Q_2 - \Omega_{23}Q_1)(\Omega_8 - \Omega_5) + (\Omega_{18}Q_2 + \Omega_{16}Q_1)(\Omega_7 - \Omega_6)\Big]$$

$$u_{r\,(x=0)} = -\frac{1}{2\beta^3 D}\left(\Omega_{18} Q_2 + \Omega_{16} Q_1\right)$$

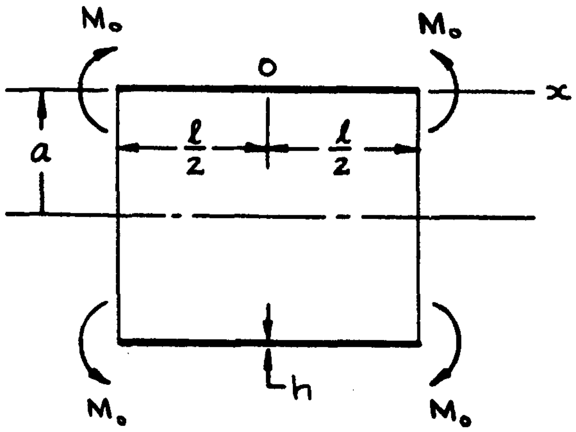
$$\theta_{(x=0)} = -\frac{1}{2\beta^2 D}\left(2\Omega_{21} Q_2 - \Omega_{20} Q_1\right)$$

$$u_{r\,(x=\ell)} = -\frac{1}{2\beta^3 D}\left(\Omega_{16} Q_2 + \Omega_{18} Q_1\right)$$

$$\theta_{(x=\ell)} = -\frac{1}{2\beta^2 D}\left(\Omega_{20} Q_2 - 2\Omega_{21} Q_1\right)$$

3.1.d Short Circular Cylindrical Shell Bent by Moments Distributed Along the Edges

1) Equal Edge Moments



$$\sigma_x = \pm \frac{12M_0}{h^2}\left[(\Omega_{11}+\Omega_{10})\Omega_8 - (\Omega_{11}-\Omega_{10})\Omega_5\right]$$

$$\sigma_\theta = \frac{4M_0}{h}\left\{\beta^2 a\left[\Omega_{11}(\Omega_5+\Omega_8)+\Omega_{10}(\Omega_5-\Omega_8) \pm \frac{3\nu}{h}\left[\Omega_{10}(\Omega_5+\Omega_8)-\Omega_{11}(\Omega_5-\Omega_8)\right]\right]\right\}$$

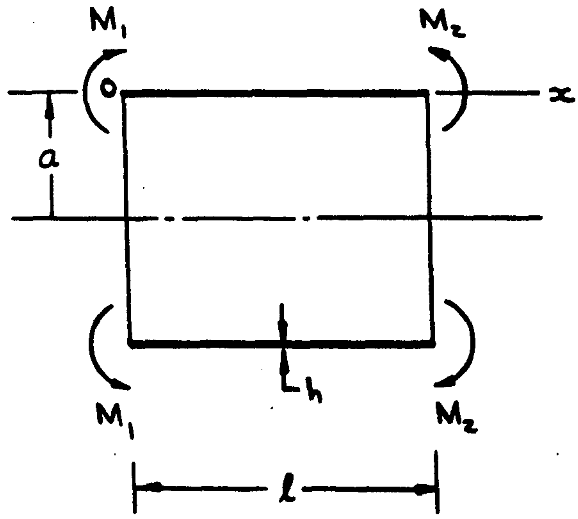
$$u_r = -\frac{M_0}{\beta^2 D}\left[(\Omega_{11}+\Omega_{10})\Omega_5 + (\Omega_{11}-\Omega_{10})\Omega_8\right]$$

$$\theta = -\frac{2M_0}{\beta D}(\Omega_{11}\Omega_7 + \Omega_{10}\Omega_6)$$

$$u_{r\,(x=\pm\frac{l}{2})} = -\frac{1}{2\beta^2 D} M_0 \Omega_{13}$$

$$\theta_{\,(x=\pm\frac{l}{2})} = \mp \frac{1}{\beta D} M_0 \Omega_{15}$$

2) **Unequal Edge Moments**



$$\sigma_x = \pm \frac{6}{h^2}\Big[(M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 - \Omega_7) - (M_1\Omega_{20} - 2M_2\Omega_{21})\Omega_5 + M_1\Omega_8\Big]$$

$$\sigma_\theta = \frac{2}{h}\Big\{\beta^2 a\Big[M_1\Omega_5 - (M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 + \Omega_7) + \Omega_8(M_1\Omega_{20} - 2M_2\Omega_{21})\Big]$$
$$\pm \frac{3\nu}{h}\Big[(M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 - \Omega_7) - \Omega_5(M_1\Omega_{20} - 2M_2\Omega_{21}) + M_1\Omega_8\Big]\Big\}$$

$$u_r = \frac{1}{2\beta^2 D}\Big[(M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 + \Omega_7) - (M_1\Omega_{20} - 2M_2\Omega_{21})\Omega_8 - M_1\Omega_5\Big]$$

$$\theta = \frac{1}{2\beta D}\Big[2\Omega_8(M_1\Omega_{17} - M_2\Omega_{19}) - (M_1\Omega_{20} - 2M_2\Omega_{21})(\Omega_7 - \Omega_6)$$
$$- M_1(\Omega_6 + \Omega_7)\Big]$$

$$u_{r(x=0)} = -\frac{1}{2\beta^2 D}\Big[M_1\Omega_{20} - 2M_2\Omega_{21}\Big]$$

$$\theta_{(x=0)} = \frac{1}{\beta D}(M_1\Omega_{17} - M_2\Omega_{19})$$

$$u_{r(x=\ell)} = \frac{1}{2\beta^2 D}(2M_1\Omega_{21} - M_2\Omega_{20})$$

$$\theta_{(x=\ell)} = -\frac{1}{\beta D}(M_1\Omega_{19} - M_2\Omega_{17})$$

3.2 Shells of Surfaces of Revolution

The following signs convention for the rotation and deflection are used:

Positive Deflection - radially inward with respect to the axis of revolution of the shell

Positive Rotation - clockwise viewing the upper cut of the shell

The nomenclature used in this section is:

σ_ϕ = meridional stress

σ_θ = circumferential stress

u_r = radial displacement

θ = slope

ϕ = angle between axis of revolution and normal to wall

R = mean radius of circumference of discontinuity circle

R_1 = radius of curvature of the section perpendicular to the meridian at the point in question

R_2 = radius of curvature of the meridional section

$$\beta^4 = \frac{3(1-\nu^2)}{R_1^2 h^2}$$

D = flexmal rigidity of the shell

$$= \frac{Eh^3}{12(1-\nu^2)}$$

Q_o = uniformly distributed circumferential shearing force at the shell edge perpendicular to the meridian

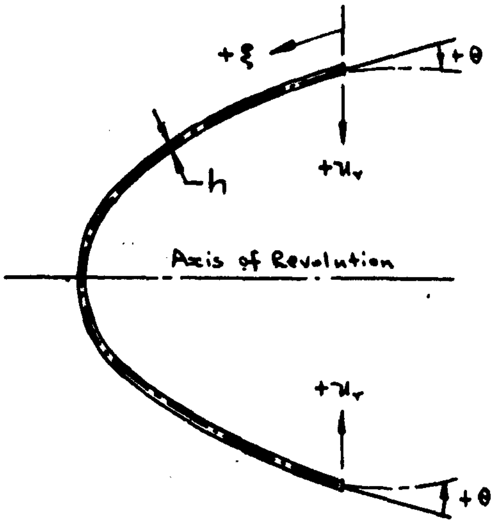
M_o = uniformly distributed circumferential bending moment at the shell edge

ξ = distance measured along the meridian of the shell from the edge to the point in question and is positive away from the junction.

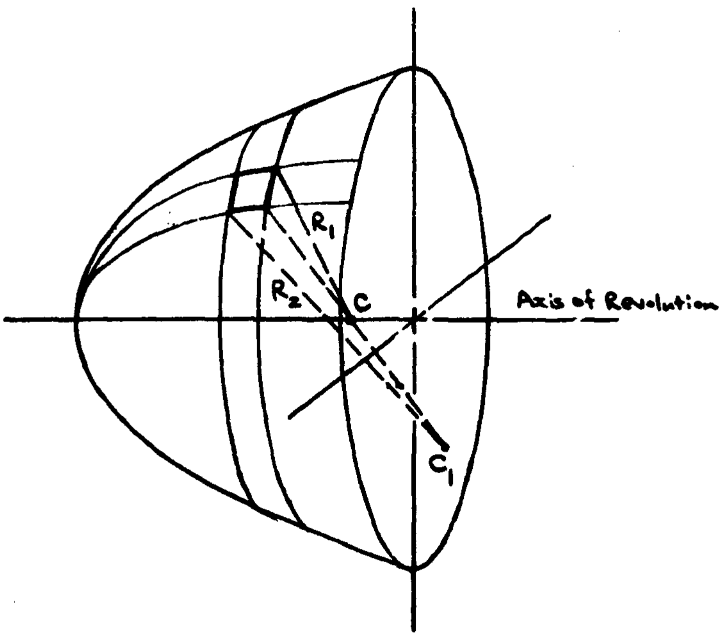
Σ = summation

ν = Poisson's ratio

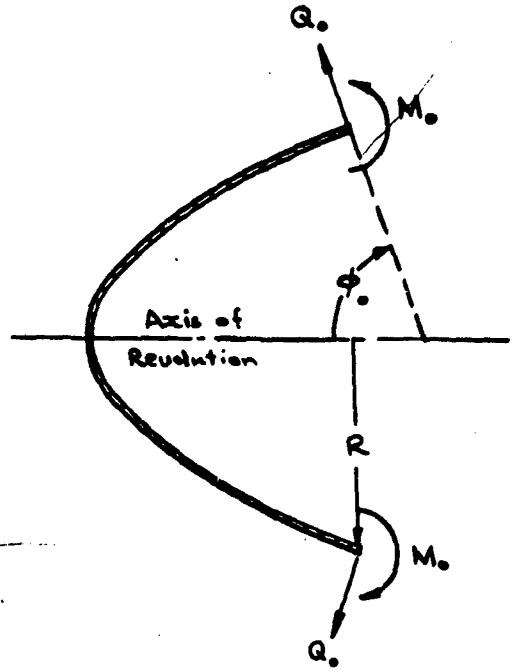
The signs convention and the pertinent notations are shown below:



Signs Convention and Notations



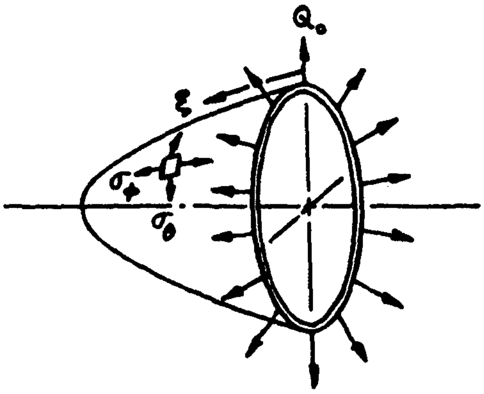
Principal Radii of Curvature of Shell



Notations in Shell Analysis

Where ± signs occur in the expressions for stresses, the upper sign refers to the inner surface of the shell and the lower sign to the outer surface of the shell.

3. 2. a Uniform Normal Shear Q_o Lb Per Linear Inch of Circumference at End



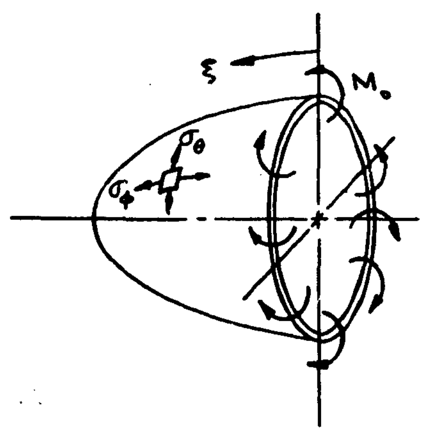
$$\sigma_\phi = \pm \frac{6RQ_o}{h^2 R_1 \beta}\, \Omega_1$$

$$\sigma_\theta = \frac{2Q_o}{h}\left\{\frac{\beta R^2}{R_1}\, \Omega_2 \pm \frac{3}{R_1 \beta h}\left[\nu R \Omega_1 - \frac{(1-\nu^2)\cot \phi_o}{2\beta}\, \Omega_3\right]\right\}$$

$$u_{r(\xi=0)} = -\frac{Q_o \sin^2 \phi_o}{2\beta^3 D}\left(1 - \frac{\nu}{2\beta R_1} \cot \phi_o\right)$$

$$\theta_{(\xi=0)} = -\frac{Q_o \sin \phi_o}{2\beta^2 D}$$

3.2.b <u>Uniform Radial Moment M_o In-Lb Per Linear Inch of Circumference at End</u>



$$\sigma_\phi = \pm \frac{6RM_o}{h^2 R_1} \Omega_3$$

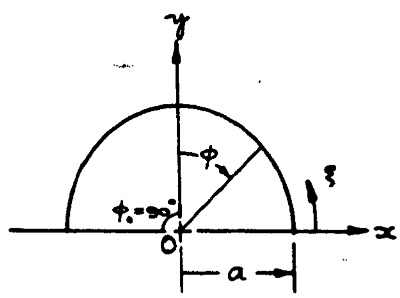
$$\sigma_\theta = \frac{2M_o}{R_1 h}\left\{\beta^2 R^2 \Omega_4 \pm \frac{3}{h}\left[\nu R \Omega_3 - \frac{(1-\nu^2)\cot\phi_o}{\beta}\Omega_2\right]\right\}$$

$$u_{r(\xi=0)} = -\frac{M_o}{2\beta^2 D}\sin\phi_o$$

$$\theta_{(\xi=0)} = -\frac{M_o}{\beta D}$$

In applying the above formulas for the determination of stresses and displacements for shells of surfaces of revolution, we note the following geometrical relationships for two of the most commonly encountered head closure designs; viz., (i) full hemispherical head closure; and (ii) full ellipsoidal head closure.

(i) Full Hemispherical Head Closure



If $\underline{a}$ is the radius to the middle surface of the hemispherical shell, we have the following geometrical relationships:

$$R_1 = R_2 = R = a$$

$$\beta^4 = \frac{3(1 - \nu^2)}{a^2 h^2}$$

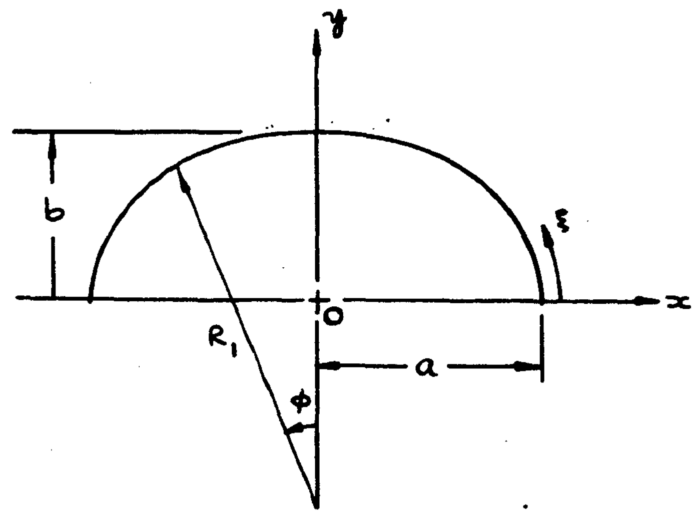
$$\xi = a\left(\frac{\pi}{2} - \phi\right)$$

$$\phi_o = 90^\circ$$

$$\sin \phi_o = 1$$

$$\cot \phi_o = 0$$

(ii) Full Ellipsoidal Head Closure



Let a and b represent the semi-major axis and semi-minor axis of the ellipsoidal head closure. Then the following geometrical relationships exist:

$$R_1 = \frac{a^2 b^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{3/2}}$$

$$= \frac{(a^4 y^2 + b^4 x^2)^{1/2}}{b^2}$$

$$R_2 = \frac{a^2}{(a^2 \sin^2 \phi + b^2 \cos^2 \phi)^{1/2}}.$$

$$= \frac{(a^4 y^2 + b^4 x^2)^{3/2}}{a^4 b^4}$$

$$R = a$$

$$\phi^o = 90^o$$

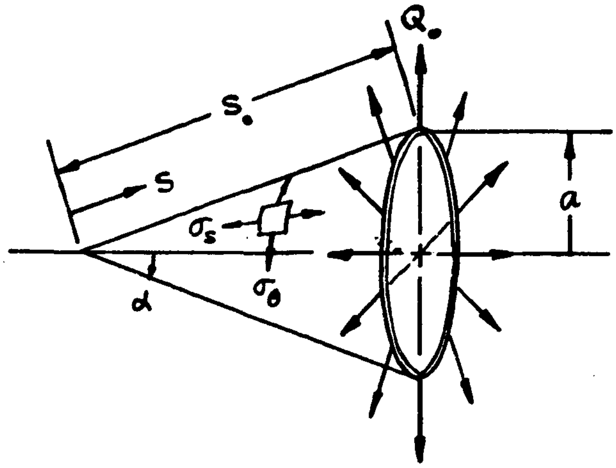
$$\sin \phi_o = 1$$

$$\cot \phi_o = 0$$

$$d\xi = R_2 d\phi$$

For the case in which the shell of surface of revolution is a right circular cone, the formulas for stresses may be more appropriately represented by the following expressions.

3.2.c Uniform Radial Shear Q_o Lb Per Linear Inch of Circumference at End of Right Circular Conical Shell



$$\sigma_s = \frac{1}{sh}\left\{C_1\left[\mathrm{ber}_2\,\xi \mp \frac{3}{m^2}\left(\xi\,\mathrm{bei}_2'\,\xi + 2\nu\,\mathrm{bei}_2\,\xi\right)\right] + C_2\left[\mathrm{bei}_2\,\xi \pm \frac{3}{m^2}\left(\xi\,\mathrm{ber}_2'\,\xi + 2\nu\,\mathrm{ber}_2\,\xi\right)\right]\right\}$$

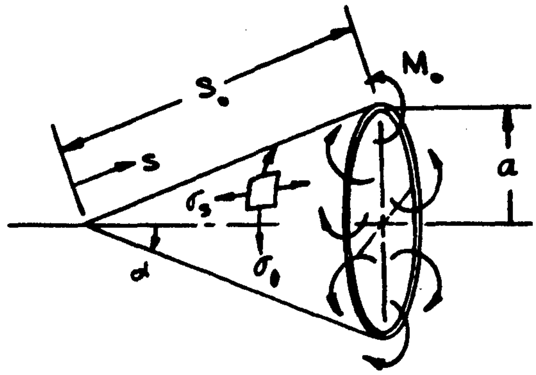
$$\sigma_\theta = \frac{1}{sh}\left\{C_1\left[\frac{1}{2}\xi\,\mathrm{ber}_2'\,\xi \mp \frac{3}{m^2}\left(2\,\mathrm{bei}_2\,\xi + \nu\,\xi\,\mathrm{bei}_2'\,\xi\right)\right] + C_2\left[\frac{1}{2}\xi\,\mathrm{bei}_2'\,\xi \pm \frac{3}{m^2}\left(2\,\mathrm{ber}_2\,\xi + \nu\,\xi\,\mathrm{ber}_2'\,\xi\right)\right]\right\}$$

where $$C_1 = \frac{Q_o s_o \sin\alpha\left(\xi_o\,\mathrm{ber}_2'\,\xi_o + 2\nu\,\mathrm{ber}_2\,\xi_o\right)}{\xi_o\left(\mathrm{ber}_2\,\xi_o\,\mathrm{ber}_2'\,\xi_o + \mathrm{bei}_2\,\xi_o\,\mathrm{bei}_2'\,\xi_o\right) + 2\nu\left[\left(\mathrm{ber}_2\,\xi_o\right)^2 + \left(\mathrm{bei}_2\,\xi_o\right)^2\right]}$$

$$C_2 = \frac{Q_o s_o \sin\alpha\left(\xi_o\,\mathrm{bei}_2'\,\xi_o + 2\nu\,\mathrm{bei}_2\,\xi_o\right)}{\xi_o\left(\mathrm{ber}_2\,\xi_o\,\mathrm{ber}_2'\,\xi_o + \mathrm{bei}_2\,\xi_o\,\mathrm{bei}_2'\,\xi_o\right) + 2\nu\left[\left(\mathrm{ber}_2\,\xi_o\right)^2 + \left(\mathrm{bei}_2\,\xi_o\right)^2\right]}$$

$$m = \sqrt[4]{12(1-\nu^2)}$$

3.2.d <u>Uniform Radial Moment M_0 In-Lb Per Linear Inch of Circumference at End of Right Circular Conical Shell</u>



$\left.\begin{matrix}\sigma_s \\ \sigma_\theta\end{matrix}\right\}$ same expressions as for case 3.2.c except

$$C_1 = -\frac{2M_0 m^2 s_0 \,\mathrm{bei}_2\,\xi_0}{h\left\{\xi_0\left(\mathrm{ber}_2\,\xi_0\,\mathrm{ber}_2'\,\xi_0 + \mathrm{bei}_2\,\xi_0\,\mathrm{bei}_2'\,\xi_0\right) + 2\nu\left[\left(\mathrm{ber}_2\,\xi_0\right)^2 + \left(\mathrm{bei}_2\,\xi_0\right)^2\right]\right\}}$$

$$C_2 = \frac{2M_0 m^2 s_0 \,\mathrm{ber}_2\,\xi_0}{h\left\{\xi_0\left(\mathrm{ber}_2\,\xi_0\,\mathrm{ber}_2'\,\xi_0 + \mathrm{bei}_2\,\xi_0\,\mathrm{bei}_2'\,\xi_0\right) + 2\nu\left[\left(\mathrm{ber}_2\,\xi_0\right)^2 + \left(\mathrm{bei}_2\,\xi_0\right)^2\right]\right\}}$$

3.2.e Bending Stresses in Cone Loaded by Uniform Internal Pressure p

When there are no edge shears or moments, but the edge support condition is such that it is not the same as that necessary for the membrane condition, bending stresses will be introduced. The magnitudes of these stresses may be obtained from the following equations:

$$\sigma_s = \frac{1}{sh}\left\{C_1\left[\operatorname{ber}_2\xi \mp \frac{3}{m^2}\left(\xi\operatorname{bei}_2'\xi + 2\nu\operatorname{bei}_2\xi\right)\right] + C_2\left[\operatorname{bei}_2\xi \pm \frac{3}{m^2}\left(\xi\operatorname{ber}_2'\xi + 2\nu\operatorname{ber}_2\xi\right)\right]\right\} \pm \frac{3\tan^2\alpha}{4(1-\nu)}p$$

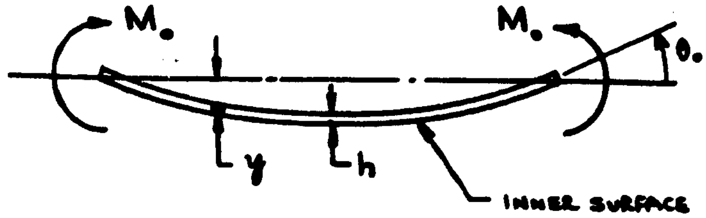
$$\sigma_\theta = \frac{1}{sh}\left\{C_1\left[\frac{1}{2}\xi\operatorname{ber}_2'\xi \mp \frac{3}{m^2}\left(2\operatorname{bei}_2\xi + \nu\xi\operatorname{bei}_2'\xi\right)\right] + C_2\left[\frac{1}{2}\xi\operatorname{bei}_2'\xi \pm \frac{3}{m^2}\left(2\operatorname{ber}_2\xi + \nu\xi\operatorname{ber}_2'\xi\right)\right]\right\} \pm \frac{3\tan^2\alpha}{4(1-\nu)}p$$

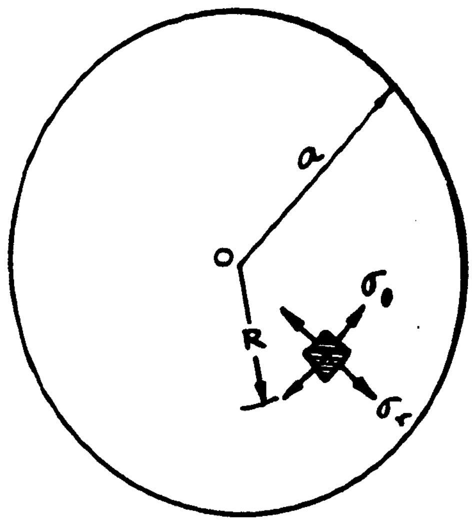
where

$$C_1 = -\frac{ps_0\tan\alpha\left[2s_0\sin^2\alpha\left(\xi_0\operatorname{ber}_2'\xi_0 + 2\nu\operatorname{ber}_2\xi_0\right) - \dfrac{m^2h\tan\alpha}{1-\nu}\operatorname{bei}_2\xi_0\right]}{4\left\{\xi_0\left(\operatorname{ber}_2\xi_0\operatorname{ber}_2'\xi_0 + \operatorname{bei}_2\xi_0\operatorname{bei}_2'\xi_0\right) + 2\nu\left[\left(\operatorname{ber}_2\xi_0\right)^2 + \left(\operatorname{bei}_2\xi_0\right)^2\right]\right\}}$$

$$C_2 = -\frac{ps_0\tan\alpha\left[2s_0\sin^2\alpha\left(\xi_0\operatorname{bei}_2'\xi_0 + 2\nu\operatorname{bei}_2\xi_0\right) + \dfrac{m^2h\tan\alpha}{1-\nu}\operatorname{ber}_2\xi_0\right]}{4\left\{\xi_0\left(\operatorname{ber}_2\xi_0\operatorname{ber}_2'\xi_0 + \operatorname{bei}_2\xi_0\operatorname{bei}_2'\xi_0\right) + 2\nu\left[\left(\operatorname{ber}_2\xi_0\right)^2 + \left(\operatorname{bei}_2\xi_0\right)^2\right]\right\}}$$

3.3. Circular Plate with Uniform Edge Moment M_o In-Lb Per Unit Inch of Circumference





At Any Point:

$$\left.\begin{aligned} \sigma_r &= \pm\frac{6M_o}{h^2} \\ \sigma_\theta &= \pm\frac{6M_o}{h^2} \end{aligned}\right\}$$ upper sign refers to the inner surface and lower sign to the outer surface.

At R:

$$y = \frac{6(1-\nu)(a^2 - R^2)\,M_o}{Eh^3}$$

<u>At Center:</u> $$y_{max} = \frac{6(1 - \nu) a^2 M_o}{Eh^3}$$

<u>At Edge:</u> $$\theta_o = \frac{12(1 - \nu) a M_o}{Eh^3}$$

4. SOLUTIONS FOR DISCONTINUITY SHEAR FORCES AND BENDING MOMENTS

A tabulation of solutions for discontinuity shear forces and bending moments for pressure vessel configurations of practical interest under the action of internal pressure is offered in this section. The positive sense of these discontinuity forces is defined in each of the figures associated with the particular pressure vessel configuration investigated.

The following nomenclature is used in this section:

a = mean radius of curvature of the circular cylinder, inches;

p = internal pressure, psi;

E = modulus of elasticity, psi;

ν = Poisson's ratio

Q_o = uniformly distributed circumferential shearing force at the junction, lb per inch;

M_o = uniformly distributed circumferential bending moment at the junction, in.-lb per inch

h = shell thickness, inch;

$$\beta_i^4 = \frac{3(1-\nu^2)}{a^2 h_i^2}$$

$$D_i = \frac{E h_i^3}{12(1-\nu^2)}$$

= flexmal rigidity of the shell

4.1 Two Long Cylinders of Unequal Thickness

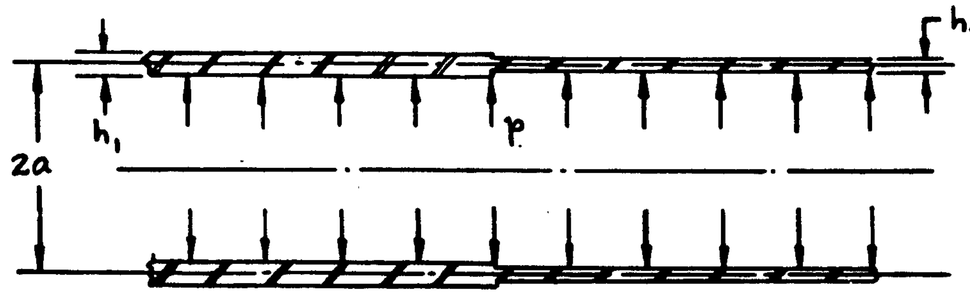


Fig. 4.1.1. Two Long Circular Cylindrical Shells of Unequal Thicknesses under Internal Pressure

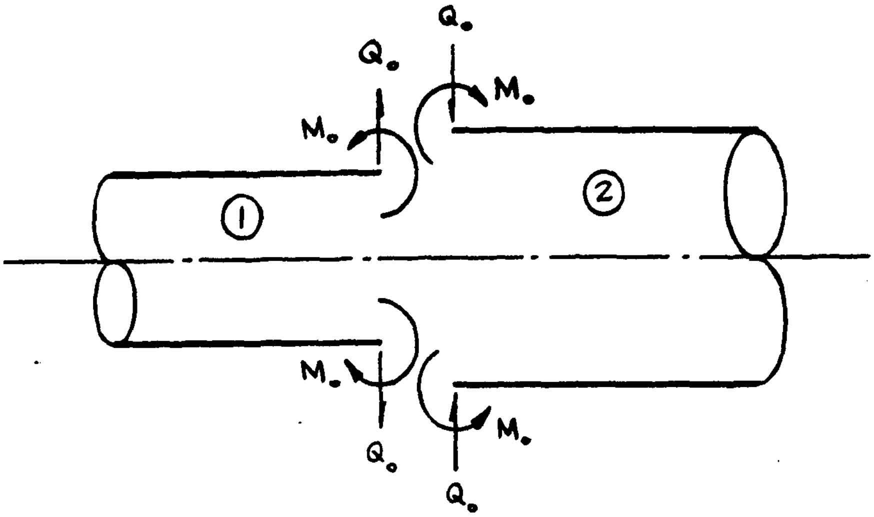


Fig. 4.1.2. Forces and Moments at the Junction of the Two Long Unequal Thickness Circular Cylindrical Shells.

$$\frac{2\beta_1 Q_o}{p(2-\nu)} = \frac{(c-1)(c^{5/2}+1)}{(c^2+1)^2 + 2c^{3/2}(c+1)} \quad ; \quad (h_1 > h_2)$$

$$\frac{4\beta_1^2 M_o}{p(2-\nu)} = \frac{(c-1)(c^2-1)}{(c^2+1)^2 + 2c^{3/2}(c+1)} \quad ; \quad (h_1 > h_2)$$

where c = thickness ratio h_1/h_2

The above equations are presented in graphical forms on Pages 59 and 60.

4.2 Two Long Cylinders of Unequal Thicknesses and Mismatch of the Middle Surfaces

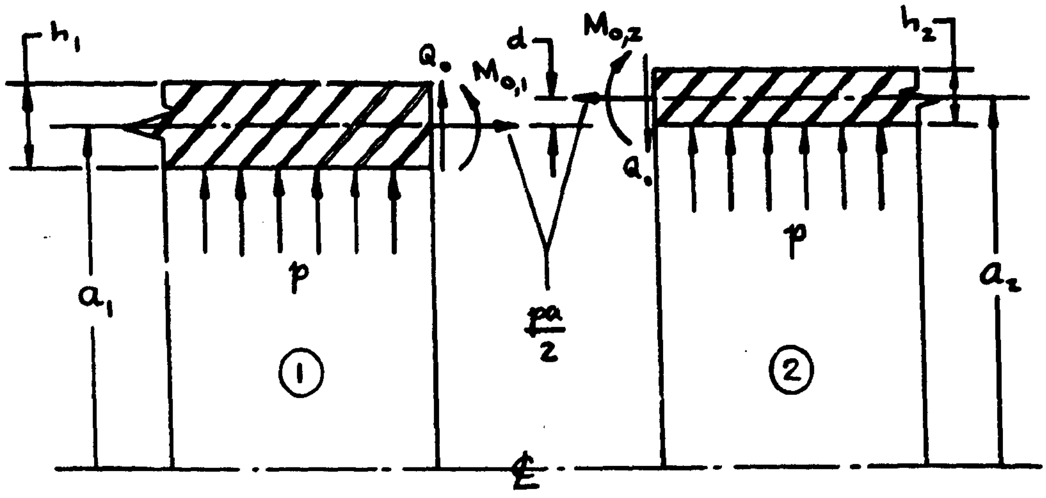


Fig. 4.2.1. Two Long Circular Cylindrical Shells of Unequal Thicknesses under Internal Pressure with Mismatch of the Middle Surface

$$Q_o = \frac{\frac{2-\nu}{2\beta_1}(c-1)(c^{5/2}+1) + da_m\beta_1 c^2(c^{1/2}+1)}{(c^2+1)^2 + 2c^{3/2}(c+1)}\,p$$

$$M_{o,1} = \frac{\frac{2-\nu}{4\beta_1^2}(c-1)(c^2-1) + \frac{1}{2}da_m c^2(c^2+2c^{1/2}+1)}{(c^2+1)^2 + 2c^{3/2}(c+1)}\,p$$

$$M_{o,2} = M_{o,1} + \frac{1}{2}dpa_m$$

where c = thickness ratio h_1/h_2

Graphs of $Q_o/da_m\beta_1 p$ and $2M_{o,1}/da_m p$ as a function of h_1/h_2 are presented on Pages 61 and 62. The Q_o and $M_{o,1}$ thus computed may be superposed to those calculated for Case 4.1.

4.3 Long Circular Cylinder with Circumferential Ring Stiffeners

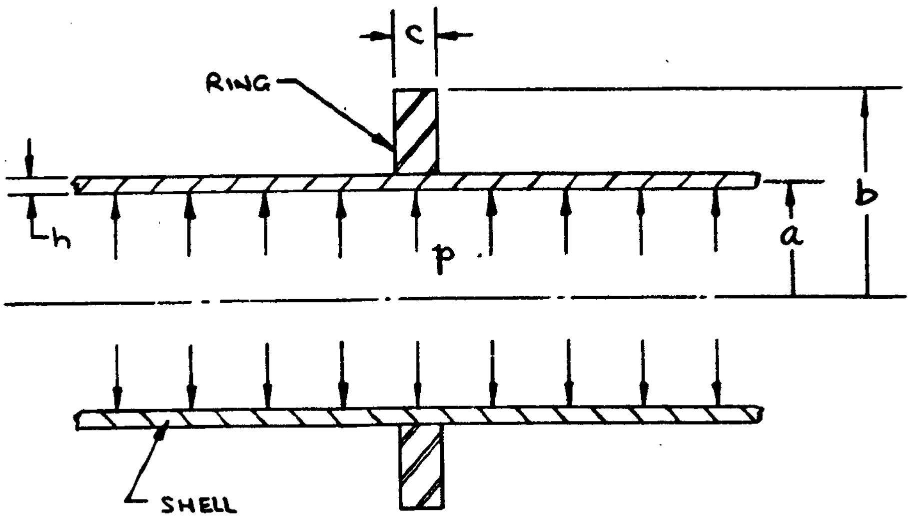


Fig. 4.3.1. Long Circular Cylinder with Circumferential Ring Stiffeners under Internal Pressure

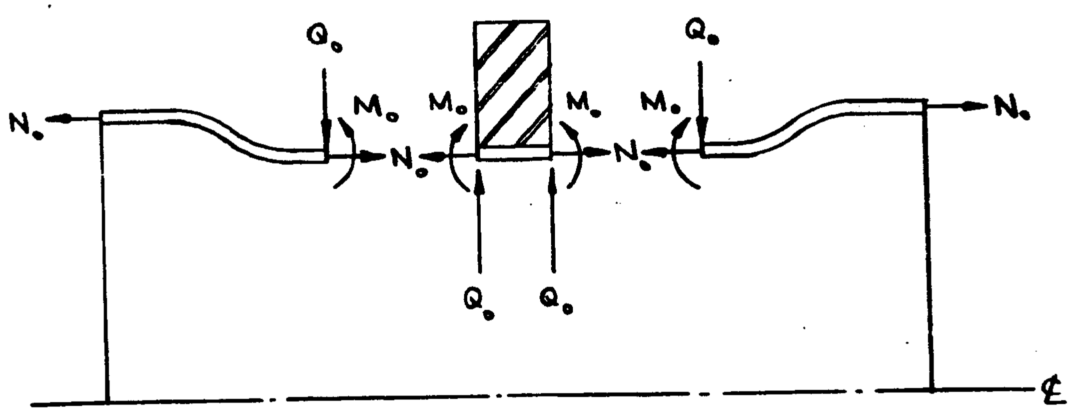


Fig. 4.3.2. Forces and Moments at the Junction of the Circular Cylindrical Shell and the Circumferential Ring Stiffener

$$M_o = \frac{p\left[\frac{a(2-\nu)}{2h} - \left(\frac{b^2+a^2}{b^2-a^2} + \nu\right)\right]}{2\beta\left[\frac{a\beta}{h} + \frac{2}{c}\left(\frac{b^2+a^2}{b^2-a^2} + \nu\right)\right]}$$

$$Q_o = 2\beta M$$

4.4 Long Circular Cylindrical Shell with Many Equidistant Circumferential Ring Stiffeners

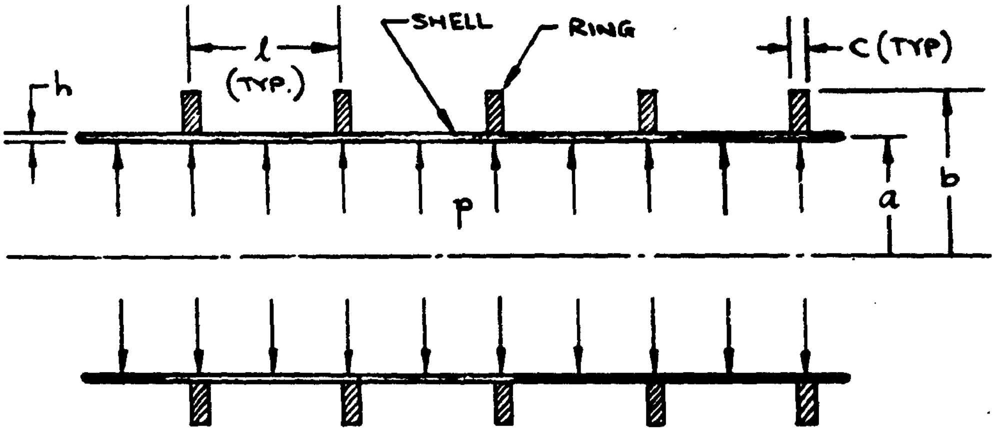


Fig. 4.4.1. Long Circular Cylindrical Shell with Many Equidistant Ring Stiffeners under Internal Pressure

$$M_o = \frac{\left[a\left(1 - \frac{\nu}{2}\right) - h\left(\frac{b^2+a^2}{b^2-a^2} + \nu\right)\right]p}{4\beta\Omega_{15}\left[\frac{h}{c}\left(\frac{b^2+a^2}{b^2-a^2} + \nu\right) + \beta a\Omega_{14}\right] - 2\beta^2 a\Omega_{13}}$$

$$Q_o = 2\beta\Omega_{15}M_o$$

where

$$\Omega_{13} = \frac{\sinh \beta \ell - \sin \beta \ell}{\sinh \beta \ell + \sin \beta \ell}$$

$$\Omega_{14} = \frac{\cosh \beta \ell + \cos \beta \ell}{\sinh \beta \ell + \sin \beta \ell}$$

$$\Omega_{15} = \frac{\cosh \beta \ell - \cos \beta \ell}{\sinh \beta \ell - \sin \beta \ell}$$

Graphs of Ω_{13}, Ω_{14} and Ω_{15} are plotted on Pages 75, 76, and 77.

4.5 Long Circular Cylindrical Shell with a Flat Head Closure

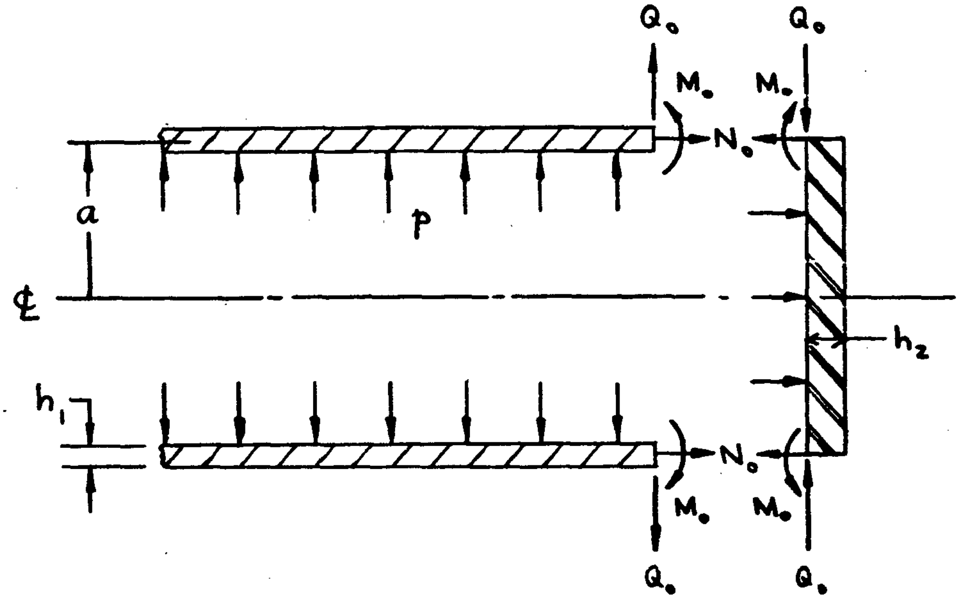


Fig. 4.5.1. Long Circular Cylindrical Shell with a Flat Head Closure under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$Q_o = 2pa\left[\frac{(a_3 - a_4)(b_1 - b_4) + b_3(a_5 - a_1)}{(a_1 - a_5)(b_2 - b_5) - (a_2 - a_6)(b_1 - b_4)}\right]$$

$$M_o = 4pa^2\left[\frac{(a_4 - a_3)(b_2 - b_5) + b_3(a_2 - a_6)}{(a_1 - a_5)(b_2 - b_5) - (a_2 - a_6)(b_1 - b_4)}\right]$$

where

$$a_1 = -\frac{6a}{h_2}(1 - \nu) \qquad b_1 = \frac{6(1 - \nu)}{\beta^2 h_1 h_2}$$

$$a_2 = 2(1 - \nu) \qquad b_2 = -\frac{3(1 - \nu)}{2\beta^2 a h_1}$$

$$a_3 = \frac{3a}{16h_2}(1 - \nu) \qquad b_3 = -\frac{3(1 - \nu)}{16\beta^2 h_1 h_2}$$

$$a_4 = -\frac{h_2}{h_1}\left(\frac{2 - \nu}{8}\right) \qquad b_4 = -2\beta a(h_2/h_1)^2$$

$$a_5 = -2\frac{h_2}{h_1}\beta^2 a^2 \qquad b_5 = -\frac{1}{2}(h_2/h_1)^2$$

$$a_6 = -\frac{h_2}{h_1}\beta a \qquad \beta^4 = \frac{3(1 - \nu^2)}{a^2 h_1^2}$$

4.6 Short Circular Cylindrical Shell with Equal Thickness Flat Head Closures

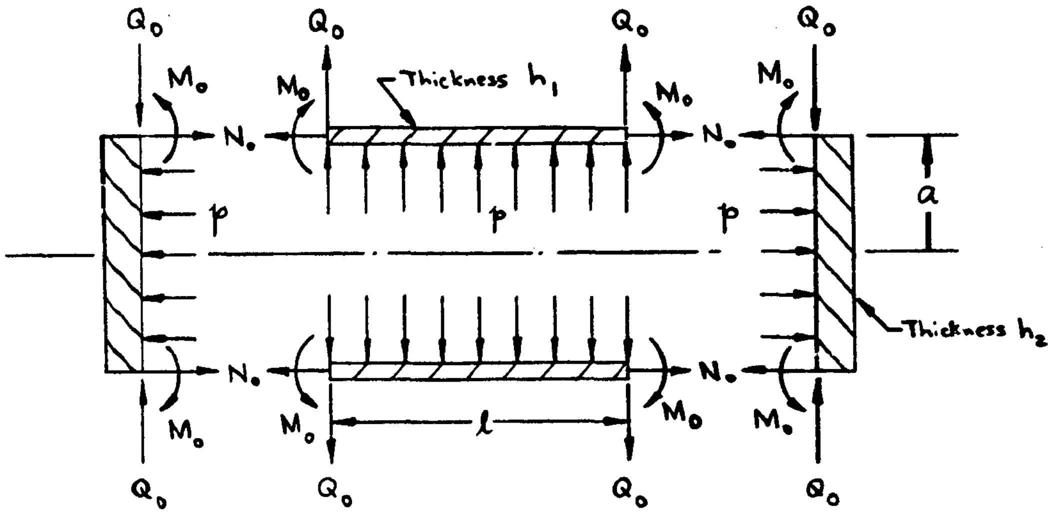


Fig. 4.6.1. Short Circular Cylindrical Shell with Equal Thickness Flat Head Closures under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$Q_o = 2pa\left[\frac{(a_3 - a_4)(b_1 - b_4\Omega_{15}) + b_3(a_5\Omega_{13} - a_1)}{(a_1 - a_5\Omega_{13})(b_2 - b_5\Omega_{13}) - (a_2 - a_6\Omega_{14})(b_1 - b_4\Omega_{15})}\right]$$

$$M_o = 4pa^2\left[\frac{(a_4 - a_3)(b_2 - b_5\Omega_{13}) + b_3(a_2 - a_6\Omega_{14})}{(a_1 - a_5\Omega_{13})(b_2 - b_5\Omega_{13}) - (a_2 - a_6\Omega_{14})(b_1 - b_4\Omega_{15})}\right]$$

where a_1 through a_6, b_1 through b_5, and β have been previously defined (Case 4.5) and Ω_{13}, Ω_{14}, and Ω_{15} are defined in Case 4.4.

4.7 Short Circular Cylindrical Shell with Unequal Thickness Flat Head Closures

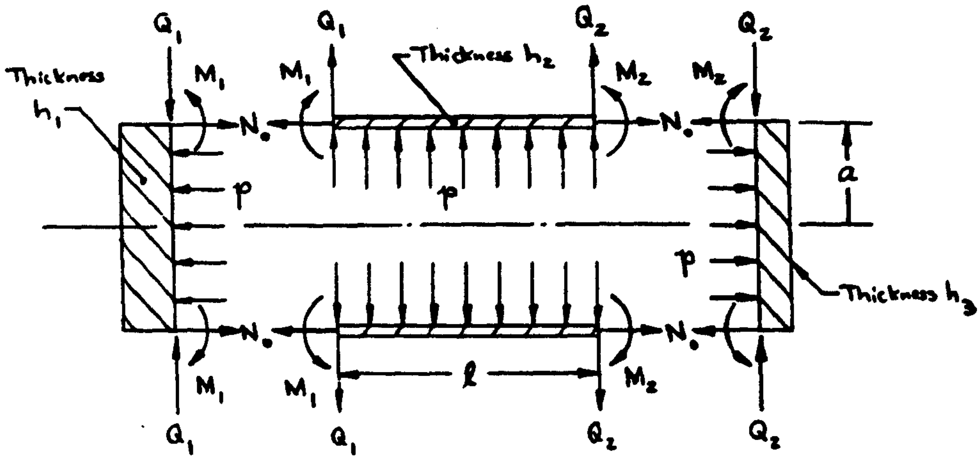


Fig. 4.7.1. Short Circular Cylindrical Shell with Unequal Thickness Flat Head Closures under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$
\begin{bmatrix} M_1 \\ M_2 \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \dfrac{A_2\Omega_{20}-A_{11}}{4pa^2} & -\dfrac{A_2\Omega_{21}}{2pa^2} & \dfrac{A_3\Omega_{16}-A_{12}}{2pa} & \dfrac{A_3\Omega_{18}}{2pa} \\ -\dfrac{A_5\Omega_{21}}{2pa^2} & \dfrac{A_5\Omega_{20}-A_{14}}{4pa^2} & \dfrac{A_6\Omega_{18}}{2pa} & \dfrac{A_6\Omega_{16}-A_{15}}{2pa} \\ \dfrac{A_7\Omega_{17}-A_{17}}{4pa^2} & -\dfrac{A_7\Omega_{19}}{4pa^2} & \dfrac{A_8\Omega_{20}-A_{18}}{2pa} & -\dfrac{A_8\Omega_{21}}{pa} \\ -\dfrac{A_9\Omega_{19}}{4pa^2} & \dfrac{A_9\Omega_{17}-A_{20}}{4pa^2} & -\dfrac{A_{10}\Omega_{21}}{pa} & \dfrac{A_{10}\Omega_{20}-A_{21}}{2pa} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{13}-A_1 \\ A_{16}-A_4 \\ A_{19} \\ A_{22} \end{bmatrix}
$$

where

$$A_1 = -\frac{h_1}{h_2}\left(\frac{2-\nu}{8}\right)$$

$$A_2 = -2\frac{h_1}{h_2}\beta^2 a^2$$

$$A_3 = -\frac{h_1}{h_2}\beta a$$

$$A_4 = -\frac{h_3}{h_2}\left(\frac{2-\nu}{8}\right)$$

$$A_5 = -2\frac{h_3}{h_2}\beta^2 a^2$$

$$A_6 = -\frac{h_3}{h_2}\beta a$$

$$A_7 = -2\beta a\left(\frac{h_1}{h_2}\right)^2$$

$$A_8 = -\frac{1}{2}\left(\frac{h_1}{h_2}\right)^2$$

$$A_9 = -2\beta a\left(\frac{h_3}{h_2}\right)^2$$

$$A_{10} = -\frac{1}{2}\left(\frac{h_3}{h_2}\right)^2$$

$$A_{11} = -\frac{6a}{h_1}(1-\nu)$$

$$A_{12} = 2(1-\nu)$$

$$A_{13} = \frac{3a}{16h_1}(1-\nu)$$

$$A_{14} = -\frac{6a}{h_3}(1-\nu)$$

$$A_{15} = 2(1-\nu)$$

$$A_{16} = \frac{3a}{16h_3}(1-\nu)$$

$$A_{17} = \frac{6(1-\nu)}{\beta^2 h_1 h_2}$$

$$A_{18} = -\frac{3(1-\nu)}{2\beta^2 a h_2}$$

$$A_{19} = -\frac{3(1-\nu)}{16\beta^2 h_1 h_2}$$

$$A_{20} = \frac{6(1-\nu)}{\beta^2 h_2 h_3}$$

$$A_{21} = -\frac{3(1-\nu)}{2\beta^2 a h_2}$$

$$A_{22} = -\frac{3(1-\nu)}{16\beta^2 h_2 h_3}$$

$$\Omega_{16} = \frac{\sinh \beta \ell \cosh \beta \ell - \sin \beta \ell \cos \beta \ell}{\sinh^2 \beta \ell - \sin^2 \beta \ell}$$ and is plotted on Page 78.

$$\Omega_{17} = \frac{\sinh \beta \ell \cosh \beta \ell + \sin \beta \ell \cos \beta \ell}{\sinh^2 \beta \ell - \sin^2 \beta \ell}$$ and is plotted on Page 79.

$$\Omega_{18} = \frac{\cos \beta \ell \sinh \beta \ell - \sin \beta \ell \cosh \beta \ell}{\sinh^2 \beta \ell - \sin^2 \beta \ell}$$ and is plotted on Page 80.

$$\Omega_{19} = \frac{\sin \beta \ell \cosh \beta \ell + \cos \beta \ell \sinh \beta \ell}{\sinh^2 \beta \ell - \sin^2 \beta \ell}$$ and is plotted on Page 81.

$$\Omega_{20} = \frac{\sinh^2 \beta \ell + \sin^2 \beta \ell}{\sinh^2 \beta \ell - \sin^2 \beta \ell}$$ and is plotted on Page 82.

$$\Omega_{21} = \frac{\sin \beta \ell \sinh \beta \ell}{\sinh^2 \beta \ell - \sin^2 \beta \ell}$$ and is plotted on Page 83.

4.8 Long Circular Cylindrical Shell with an Ellipsoidal Head Closure

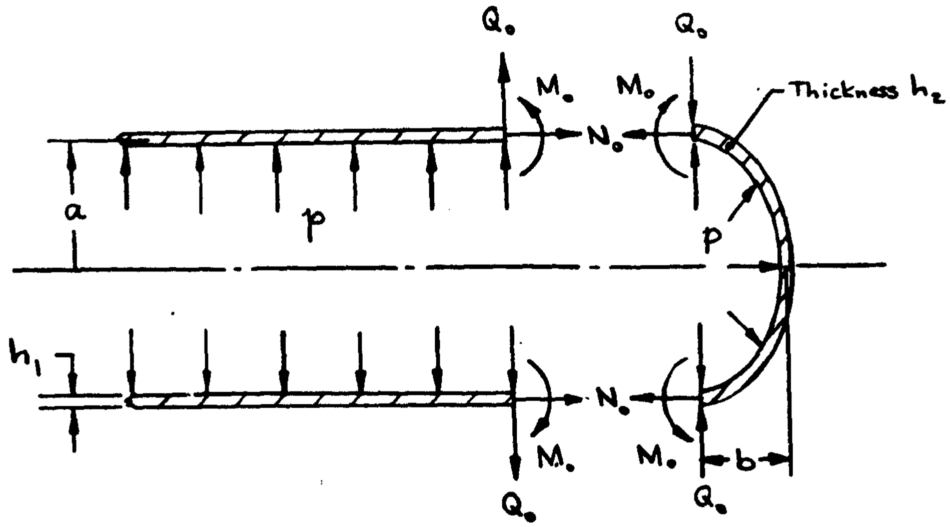


Fig. 4.8.1. Long Circular Cylindrical Shell with an Ellipsoidal Head Closure under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$\frac{Q_o}{\frac{p a^2 \beta_c^3 D_c}{E h_2}\left[(2-\nu)\left(\frac{h_2}{h_1}-1\right)+\frac{a^2}{b^2}\right]} = \frac{1+\left(\frac{h_1}{h_2}\right)^{5/2}}{\left[1+\left(\frac{h_1}{h_2}\right)^{\frac{5}{2}}\right]\left[1+\left(\frac{h_1}{h_2}\right)^{\frac{3}{2}}\right]-\frac{1}{2}\left[1-\left(\frac{h_1}{h_2}\right)^{2}\right]^{2}}$$

$$\frac{M_o}{\frac{p a^2 \beta_c^2 D_c}{2 E h_2}\left[(2-\nu)\left(\frac{h_2}{h_1}-1\right)+\frac{a^2}{b^2}\right]} = \frac{1-\left(\frac{h_1}{h_2}\right)^{2}}{\left[1+\left(\frac{h_1}{h_2}\right)^{\frac{5}{2}}\right]\left[1+\left(\frac{h_1}{h_2}\right)^{\frac{3}{2}}\right]-\frac{1}{2}\left[1-\left(\frac{h_1}{h_2}\right)^{2}\right]^{2}}$$

where $$\beta_c^4 = \frac{3(1-\nu^2)}{a^2 h_1^2}$$

$$D_c = \frac{E h_1^3}{12\,(1-\nu^2)}$$

The above expressions for Q_o and M_o are plotted as functions of h_1/h_2 on Pages 59 and 60.

4.9 Short Circular Cylindrical Shell with Equal Thickness Ellipsoidal Head Closures

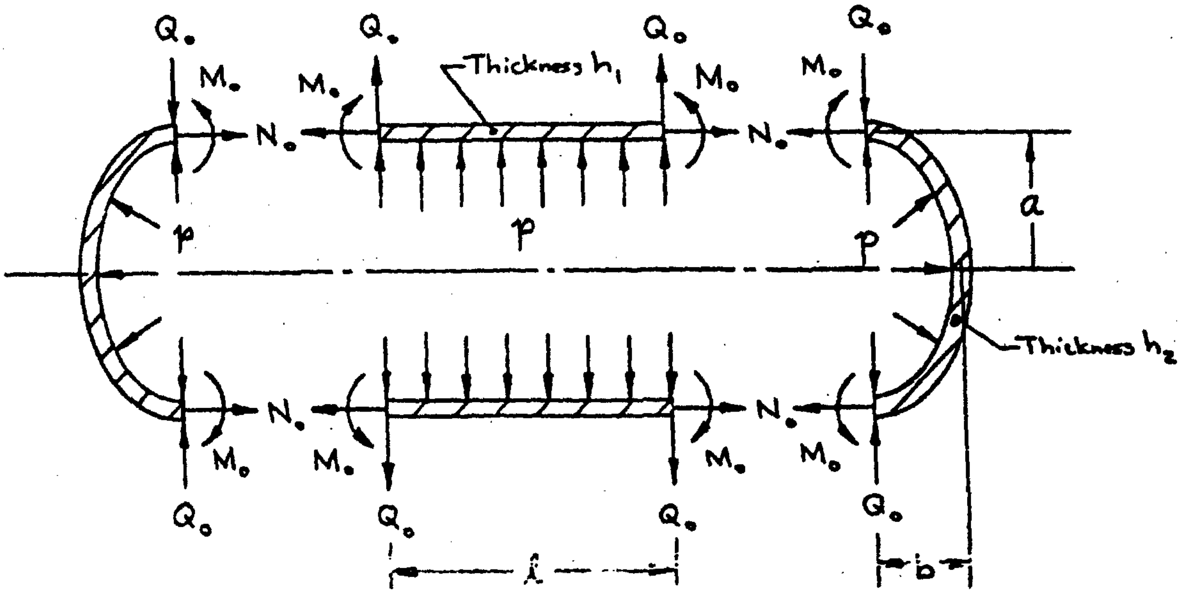


Fig. 4.9.1. Short Circular Cylindrical Shell with Equal Thickness Ellipsoidal Head Closures under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$Q_o = 2pa \left[\frac{(a_3 - a_4)(b_1 - b_4 \Omega_{15})}{(a_1 - a_5 \Omega_{13})(b_2 - b_5 \Omega_{13}) - (a_2 - a_6 \Omega_{14})(b_1 - b_4 \Omega_{15})} \right]$$

$$M_o = 4pa^2 \left[\frac{(a_4 - a_3)(b_2 - b_5 \Omega_{13})}{(a_1 - a_5 \Omega_{13})(b_2 - b_5 \Omega_{13}) - (a_2 - a_6 \Omega_{14})(b_1 - b_4 \Omega_{15})} \right]$$

where

$$a_1 = -\frac{2a}{h_2}\sqrt{3(1-\nu^2)}$$

$$b_1 = \frac{2\beta_{e_o}}{\beta_c^2 h_1}\sqrt{3(1-\nu^2)}$$

$$a_2 = \frac{1}{\beta_{e_o} h_2}\sqrt{3(1-\nu^2)}$$

$$b_2 = -\frac{\sqrt{3(1-\nu^2)}}{2\beta_c^2 a h_1}$$

$$a_3 = -\frac{1}{8}\left(2 - \frac{a^2}{b^2} - \nu\right)$$

$$b_4 = -2\beta_c a\left(\frac{h_2}{h_1}\right)^2$$

$$a_4 = -\frac{h_2}{h_1}\left(\frac{2-\nu}{8}\right)$$

$$b_5 = -\frac{1}{2}\left(\frac{h_2}{h_1}\right)^2$$

$$a_5 = -2\,\frac{h_2}{h_1}\,\beta_c^2 a^2$$

$$\beta_c^4 = \frac{3(1-\nu^2)}{a^2 h_1^2}$$

$$a_6 = -\frac{h_2}{h_1}\,\beta_c a$$

$$\beta_{e_o}^4 = \frac{3(1-\nu^2)}{a^2 h_2^2}$$

$\Omega_{13} = \dfrac{\sinh \beta\ell - \sin \beta\ell}{\sinh \beta\ell + \sin \beta\ell}$ and is plotted on Page 75.

$\Omega_{14} = \dfrac{\cosh \beta\ell + \cos \beta\ell}{\sinh \beta\ell + \sin \beta\ell}$ and is plotted on Page 76.

$\Omega_{15} = \dfrac{\cosh \beta\ell - \cos \beta\ell}{\sinh \beta\ell - \sin \beta\ell}$ and is plotted on Page 77.

4.10 Short Circular Cylindrical Shell with Unequal Thickness Ellipsoidal Head Closures

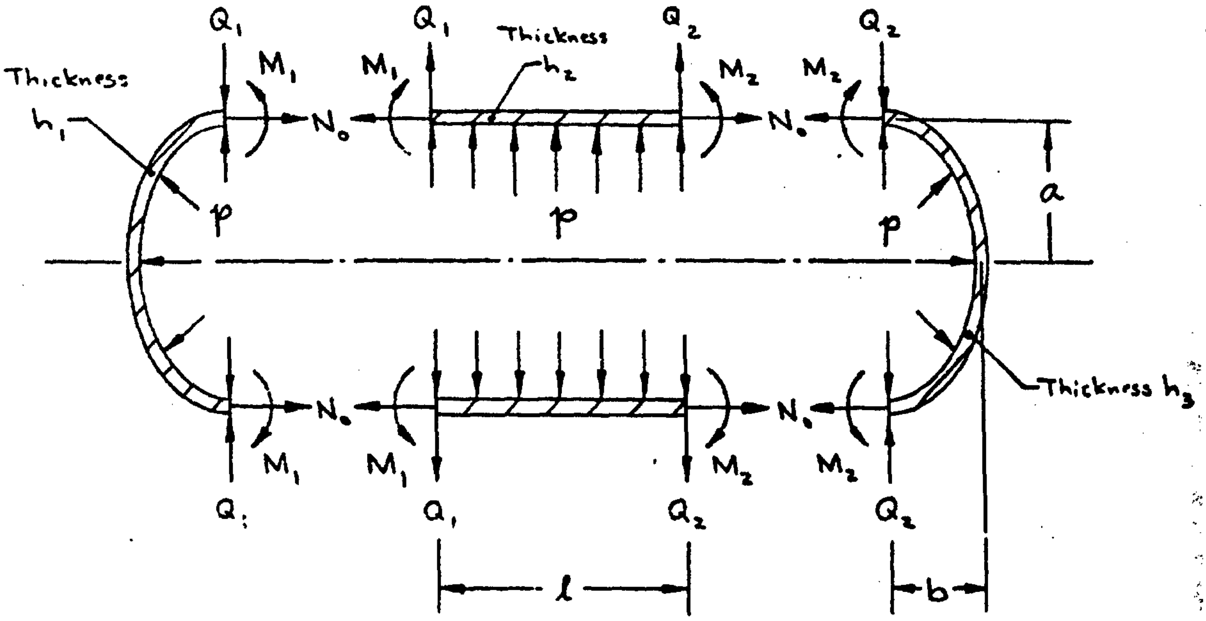


Fig. 4.10.1. Short Circular Cylindrical Shell with Unequal Thickness Ellipsoidal Head Closures under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$
\begin{bmatrix} M_1 \\ M_2 \\ Q_1 \\ Q_2 \end{bmatrix}
=
\begin{bmatrix}
\dfrac{A_2\Omega_{20} - A_{11}}{4pa^2} & -\dfrac{A_2\Omega_{21}}{2pa^2} & \dfrac{A_3\Omega_{16} - A_{12}}{2pa} & \dfrac{A_3\Omega_{18}}{2pa} \\
-\dfrac{A_5\Omega_{21}}{2pa^2} & \dfrac{A_5\Omega_{20} - A_{14}}{4pa^2} & \dfrac{A_6\Omega_{18}}{2pa} & \dfrac{A_6\Omega_{16} - A_{15}}{2pa} \\
\dfrac{A_7\Omega_{17} - A_{17}}{4pa^2} & -\dfrac{A_7\Omega_{19}}{4pa^2} & \dfrac{A_8\Omega_{20} - A_{18}}{2pa} & -\dfrac{A_8\Omega_{21}}{pa} \\
-\dfrac{A_9\Omega_{19}}{4pa^2} & \dfrac{A_9\Omega_{17} - A_{19}}{4pa^2} & -\dfrac{A_{10}\Omega_{21}}{pa} & \dfrac{A_{10}\Omega_{20} - A_{20}}{2pa}
\end{bmatrix}^{-1}
\bullet
\begin{bmatrix} A_{13} - A_1 \\ A_{16} - A_4 \\ 0 \\ 0 \end{bmatrix}
$$

where

$$A_1 = -\frac{h_1}{h_2}\left(\frac{2-\nu}{8}\right)$$

$$A_2 = -2\frac{h_1}{h_2}\beta_c^2 a^2$$

$$A_3 = -\frac{h_1}{h_2}\beta_c a$$

$$A_4 = -\frac{h_3}{h_2}\left(\frac{2-\nu}{8}\right)$$

$$A_5 = -2\frac{h_3}{h_2}\beta_c^2 a^2$$

$$A_6 = -\frac{h_3}{h_2}\beta_c a$$

$$A_7 = -2\beta_c a\left(\frac{h_1}{h_2}\right)^2$$

$$A_8 = -\frac{1}{2}\left(\frac{h_1}{h_2}\right)^2$$

$$A_9 = -2\beta_c a\left(\frac{h_3}{h_2}\right)^2$$

$$A_{10} = -\frac{1}{2}\left(\frac{h_3}{h_2}\right)^2$$

$$A_{11} = -\frac{2a}{h_1}\sqrt{3(1-\nu^2)}$$

$$A_{12} = \frac{1}{\beta_{10}h_1}\sqrt{3(1-\nu^2)}$$

$$A_{13} = -\frac{1}{8}\left(2-\frac{a^2}{b^2}-\nu\right)$$

$$A_{14} = -\frac{2a}{h_3}\sqrt{3(1-\nu^2)}$$

$$A_{15} = \frac{1}{\beta_{30}h_3}\sqrt{3(1-\nu^2)}$$

$$A_{16} = -\frac{1}{8}\left(2-\frac{a^2}{b^2}-\nu\right)$$

$$A_{17} = \frac{2\beta_{10}}{\beta_c^2 h_2}\sqrt{3(1-\nu^2)}$$

$$A_{18} = -\frac{\sqrt{3(1-\nu^2)}}{2\beta_c^2 a h_2}$$

$$A_{19} = \frac{2\beta_{30}}{\beta_c^2 h_2}\sqrt{3(1-\nu^2)}$$

$$A_{20} = -\frac{\sqrt{3(1-\nu^2)}}{2\beta_c^2 a h_2}$$

$$\beta_{10}^4 = \frac{3(1-\nu^2)}{a^2 h_1^2}$$

$$\beta_{30}^4 = \frac{3(1-\nu^2)}{a^2 h_3^2}$$

The functions Ω_{16} through Ω_{21} are identically defined as those for Case 4.7.

4.11 Long Circular Cylindrical Shell with a Hemispherical Head Closure

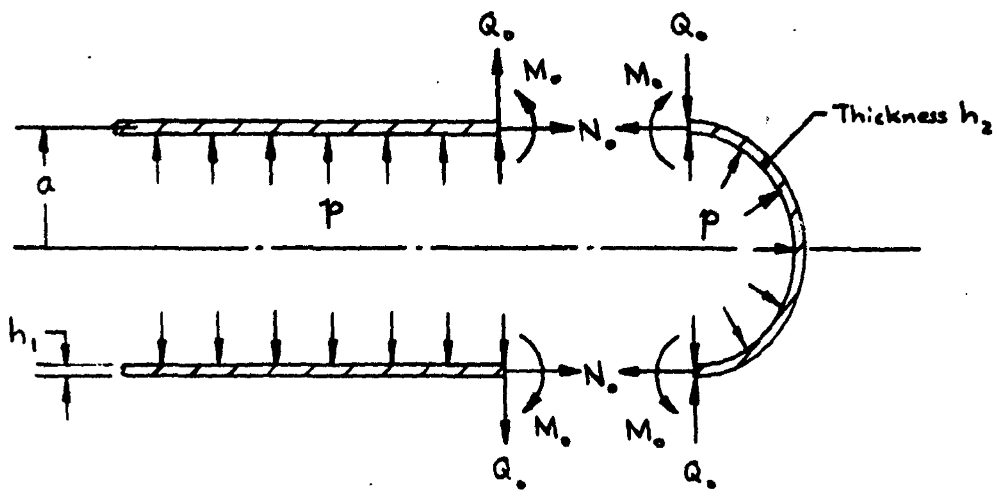


Fig. 4.11.1. Long Circular Cylindrical Shell with a Hemispherical Head Closure under Internal Pressure. The forces and moments are shown in the positive sense above.

$$\frac{M_o}{\dfrac{p a^2 \beta_c^2 D_c}{2 E h_2}\left[\dfrac{h_2}{h_1}(2-\nu)-(1-\nu)\right]} = \frac{1-\left(\dfrac{h_1}{h_2}\right)^2}{\left[1+\left(\dfrac{h_1}{h_2}\right)^{\frac{5}{2}}\right]\left[1+\left(\dfrac{h_1}{h_2}\right)^{\frac{3}{2}}\right]-\dfrac{1}{2}\left[1-\left(\dfrac{h_1}{h_2}\right)^2\right]^2}$$

$$\frac{Q_o}{\dfrac{p a^2 \beta_c^3 D_c}{E h_2}\left[\dfrac{h_2}{h_1}(2-\nu)-(1-\nu)\right]} = \frac{1+\left(\dfrac{h_1}{h_2}\right)^{5/2}}{\left[1+\left(\dfrac{h_1}{h_2}\right)^{\frac{5}{2}}\right]\left[1+\left(\dfrac{h_1}{h_2}\right)^{\frac{3}{2}}\right]-\dfrac{1}{2}\left[1-\left(\dfrac{h_1}{h_2}\right)^2\right]^2}$$

where $\beta_c^4 = \dfrac{3(1-\nu^2)}{a^2 h_1^2}$; $D_c = \dfrac{E h_1^3}{12(1-\nu^2)}$.

4.12 Short Circular Cylindrical Shell with Equal Thickness Hemispherical Head Closures

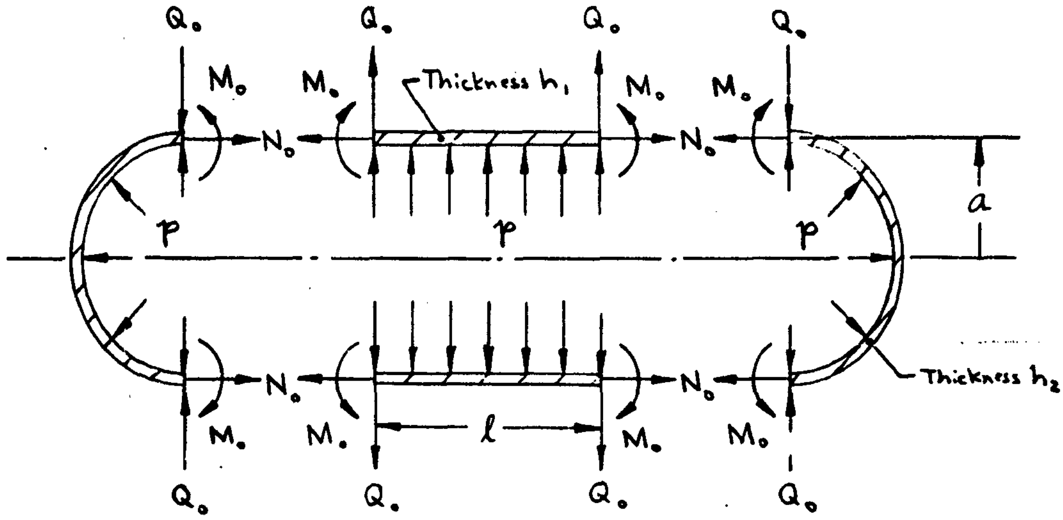


Fig. 4.12.1. Short Circular Cylindrical Shell with Equal Thickness Hemispherical Head Closures under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

The formulas for Q_o and M_o for this case are identical to those for Case 4.9. The dimensionless influence numbers a_1, a_2, ..., b_1, ... b_5 etc., are also identical with the exception of a_3, where for this case,

$$a_3 = -\frac{1}{8}(1 - \nu)$$

4.13 Short Circular Cylindrical Shell with Unequal Thickness Hemispherical Head Closures

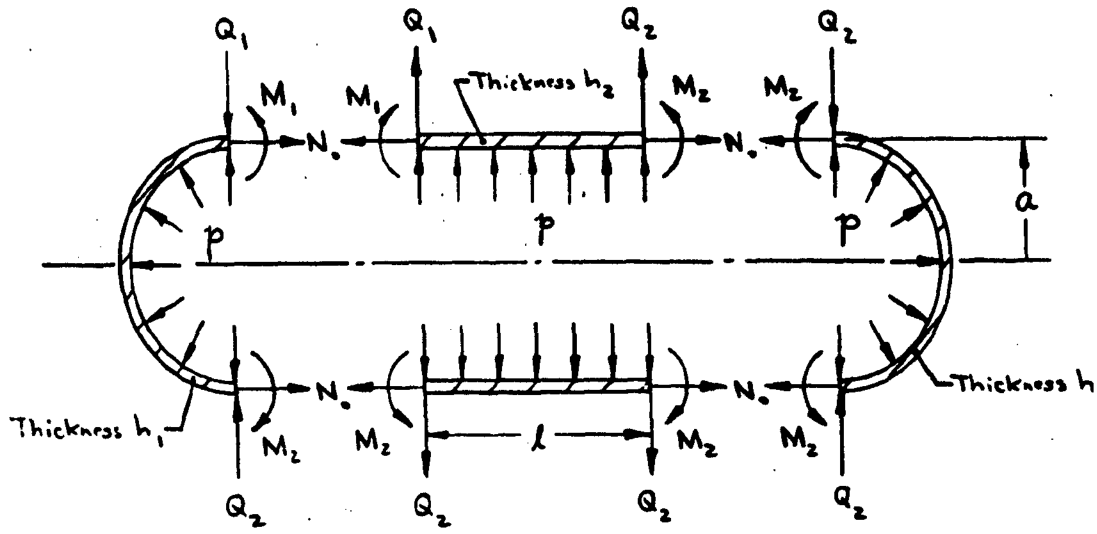


Fig. 4.13.1. Short Circular Cylindrical Shell with Unequal Thickness Hemispherical Head Closures under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

The formulas for Q_1, Q_2, M_1, and M_2 for this case are identical to those for Case 4.10. The dimensionless influence numbers A_{13} and A_{16} must, however, be changed to the following

$$A_{13} = A_{16} = -\frac{1}{8}(1 - \nu)$$

4.14 Long Circular Cylindrical Shell with a Conical Head Closure

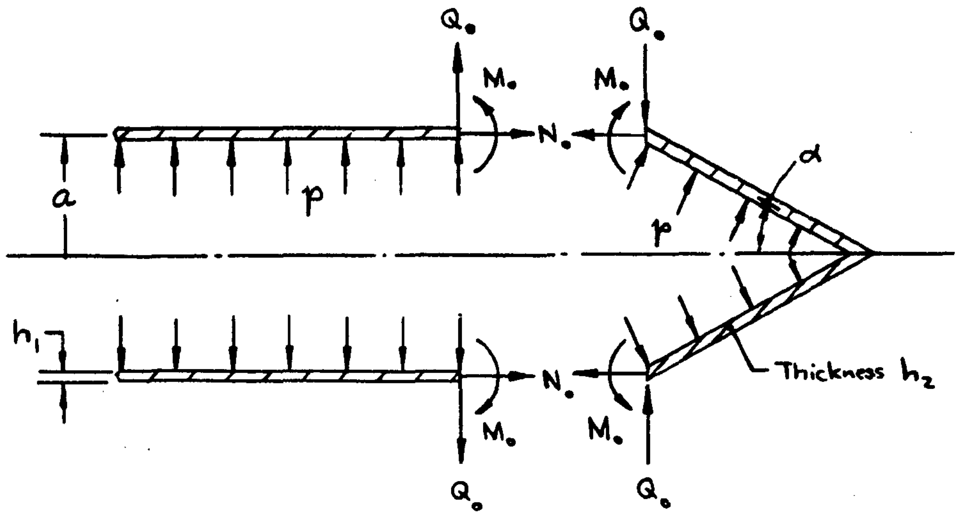


Fig. 4.14.1. Long Circular Cylindrical Shell with a Conical Head Closure under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$Q_0 = 2pa\left[\frac{b_3(a_1+a_4)+(a_3+a_6)(b_4-b_1)}{(a_1+a_4)(b_5-b_2)+(b_1-b_4)(a_2+a_5)}\right]$$

$$M_0 = 4pa^2\left[\frac{(a_3+a_6)(b_2-b_5)-b_3(a_2+a_5)}{(a_1+a_4)(b_5-b_2)+(b_1-b_4)(a_2+a_5)}\right]$$

where
$$a_1 = -\frac{b_2\,\xi_0^2\tan\alpha\sin\alpha}{2}$$

$$a_2 = -\frac{\frac{1}{4}\xi_0^2 B - \nu^2 G}{c + 2\nu G}\sin\alpha$$

$$a_3 = \frac{2-\nu}{8}\sec\alpha + \frac{a_2}{4}\tan\alpha - \frac{3b_3(1+\nu)}{\xi_o^2}\sec\alpha$$

$$b_1 = -\frac{2G}{C+2\nu G}\,\frac{\xi_o^2\tan\alpha}{4}$$

$$b_2 = -\frac{A/2}{C+2\nu G}$$

$$b_3 = -\frac{6(1+\nu)}{\xi_o^4}\,b_1\operatorname{cosec}^2\alpha + \frac{b_2\tan\alpha}{4} - \frac{3}{2}\,\frac{\sec\alpha\operatorname{cosec}\alpha}{\xi_o^2}$$

$$A = \xi_o\left(\operatorname{ber}_2'\xi_o\operatorname{bei}_2\xi_o - \operatorname{bei}_2'\xi_o\operatorname{ber}_2\xi_o\right)$$

$$B = \left(\operatorname{ber}_2'\xi_o\right)^2 + \left(\operatorname{bei}_2'\xi_o\right)^2$$

$$C = \xi_o\left(\operatorname{ber}_2\xi_o\operatorname{ber}_2'\xi_o + \operatorname{bei}_2\xi_o\operatorname{bei}_2'\xi_o\right)$$

$$G = \left(\operatorname{ber}_2\xi_o\right)^2 + \left(\operatorname{bei}_2\xi_o\right)^2$$

$$\xi_o = 2\sqrt[4]{12(1-\nu^2)}\sqrt{\frac{a}{h_2}\cot\alpha\operatorname{cosec}\alpha}$$

$$a_4 = -2\frac{h_2}{h_1}\beta_c^2a^2$$

$$a_5 = -\frac{h_2}{h_1}\beta_c a$$

$$a_6 = -\frac{h_2}{h_1}\left(\frac{2-\nu}{8}\right)$$

$$b_4 = 2\beta_c a\left(\frac{h_2}{h_1}\right)^2$$

$$b_5 = \frac{1}{2}\left(\frac{h_2}{h_1}\right)^2$$

$$\beta_c^4 = \frac{3(1-\nu^2)}{a^2h_1^2}$$

Graphs of a_1, a_2, b_1, and b_2 as a function of ξ_0 and a for ν of 0.30 are plotted and are presented on Pages 86 through 89.

4.15 Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Ellipsoidal and Conical Shape

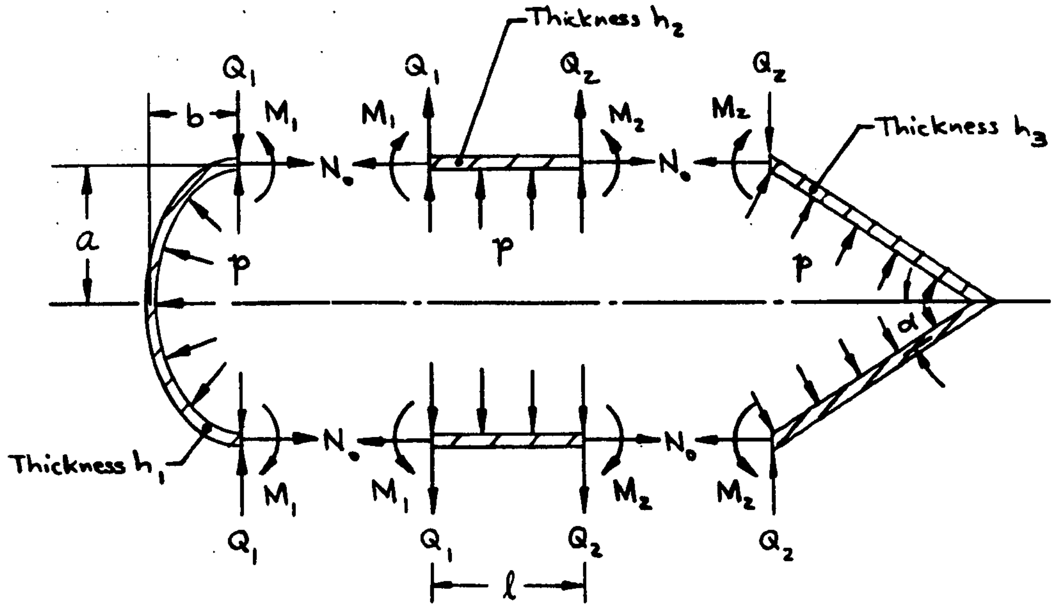


Fig. 4.15.1. Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Ellipsoidal and Conical Shape under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

$$\begin{bmatrix} M_1 \\ M_2 \\ Q_1 \\ Q_2 \end{bmatrix} = \begin{bmatrix} \frac{A_2\Omega_{20}-A_{11}}{4pa^2} & -\frac{A_2\Omega_{21}}{2pa^2} & \frac{A_3\Omega_{16}-A_{12}}{2pa} & \frac{A_3\Omega_{18}}{2pa} \\ \frac{A_7\Omega_{17}-A_{17}}{4pa^2} & -\frac{A_7\Omega_{19}}{4pa^2} & \frac{A_8\Omega_{20}-A_{18}}{2pa} & -\frac{A_8\Omega_{21}}{pa} \\ -\frac{A_5\Omega_{21}}{2pa^2} & \frac{A_5\Omega_{20}+a_1}{4pa^2} & \frac{A_6\Omega_{18}}{2pa} & \frac{A_6\Omega_{16}+a_2}{2pa} \\ -\frac{A_9\Omega_{19}}{4pa^2} & \frac{A_9\Omega_{17}+b_1}{4pa^2} & -\frac{A_{10}\Omega_{21}}{pa} & \frac{A_{10}\Omega_{20}+b_2}{2pa} \end{bmatrix}^{-1} \cdot \begin{bmatrix} A_{13}-A_1 \\ 0 \\ -a_3-A_4 \\ -b_3 \end{bmatrix}$$

where

$$A_1 = -\frac{h_1}{h_2}\left(\frac{2 - \nu}{8}\right)$$

$$A_2 = -2\frac{h_1}{h_2}\beta_c^2 a^2$$

$$A_3 = -\frac{h_1}{h_2}\beta_c a$$

$$A_4 = -\frac{h_3}{h_2}\left(\frac{2 - \nu}{8}\right)$$

$$A_5 = -2\frac{h_3}{h_2}\beta_c^2 a^2$$

$$A_6 = -\frac{h_3}{h_2}\beta_c a$$

$$A_7 = -2\beta_c a\left(\frac{h_1}{h_2}\right)^2$$

$$A_8 = -\frac{1}{2}\left(\frac{h_1}{h_2}\right)^2$$

$$A_9 = -2\beta_c a\left(\frac{h_3}{h_2}\right)^2$$

$$A_{10} = -\frac{1}{2}\left(\frac{h_3}{h_2}\right)^2$$

$$A_{11} = -\frac{2a}{h_1}\sqrt{3(1 - \nu^2)}$$

$$A_{12} = \frac{1}{\beta_{10}h_1}\sqrt{3(1 - \nu^2)}$$

$$A_{13} = -\frac{1}{8}\left(2 - \frac{a^2}{b^2} - \nu\right)$$

$$\beta_c^4 = \frac{3(1 - \nu^2)}{a^2 h_2^2}$$

$$A_{17} = \frac{2\beta_{10}}{\beta_c^2 h_2}\sqrt{3(1 - \nu^2)}$$

$$\beta_{10}^4 = \frac{3(1 - \nu^2)}{a^2 h_1^2}$$

$$A_{18} = -\frac{\sqrt{3(1 - \nu^2)}}{2\beta_c^2 a h_2}$$

The functions Ω_{16} through Ω_{21} are identical to those defined in Case 4.7.

4.16 Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Hemispherical and Conical Shape

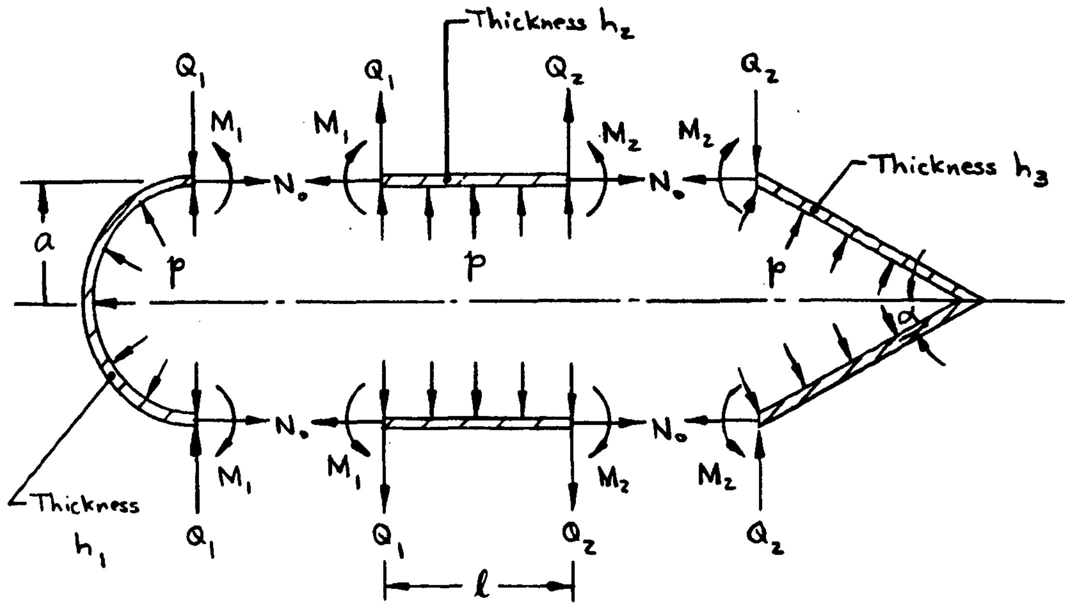


Fig. 4.16.1. Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Hemispherical and Conical Shape under Internal Pressure. The Forces and Moments are shown in the Positive Sense above.

The formulas for Q_1, Q_2, M_1, and M_2 for this configuration are identical to those for Case 4.15 except that A_{13} is now defined by

$$A_{13} = -\frac{1}{8}(1 - \nu)$$

5. GRAPHICAL PRESENTATION OF PERTINENT PARAMETERS

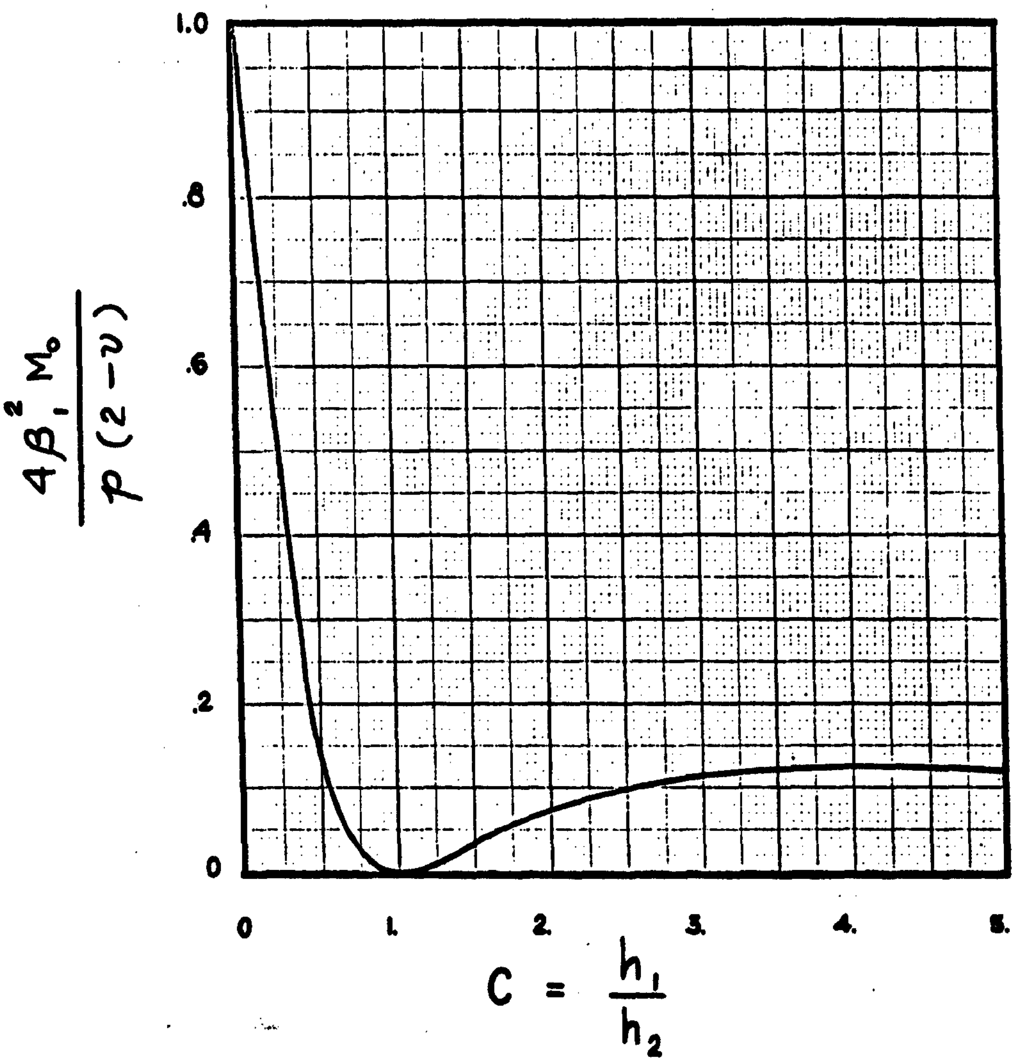


Fig. 5.1 Variation of Juncture Bending Moments with Thickness Ratio for Two Long Cylindrical Shells of Unequal Thickness

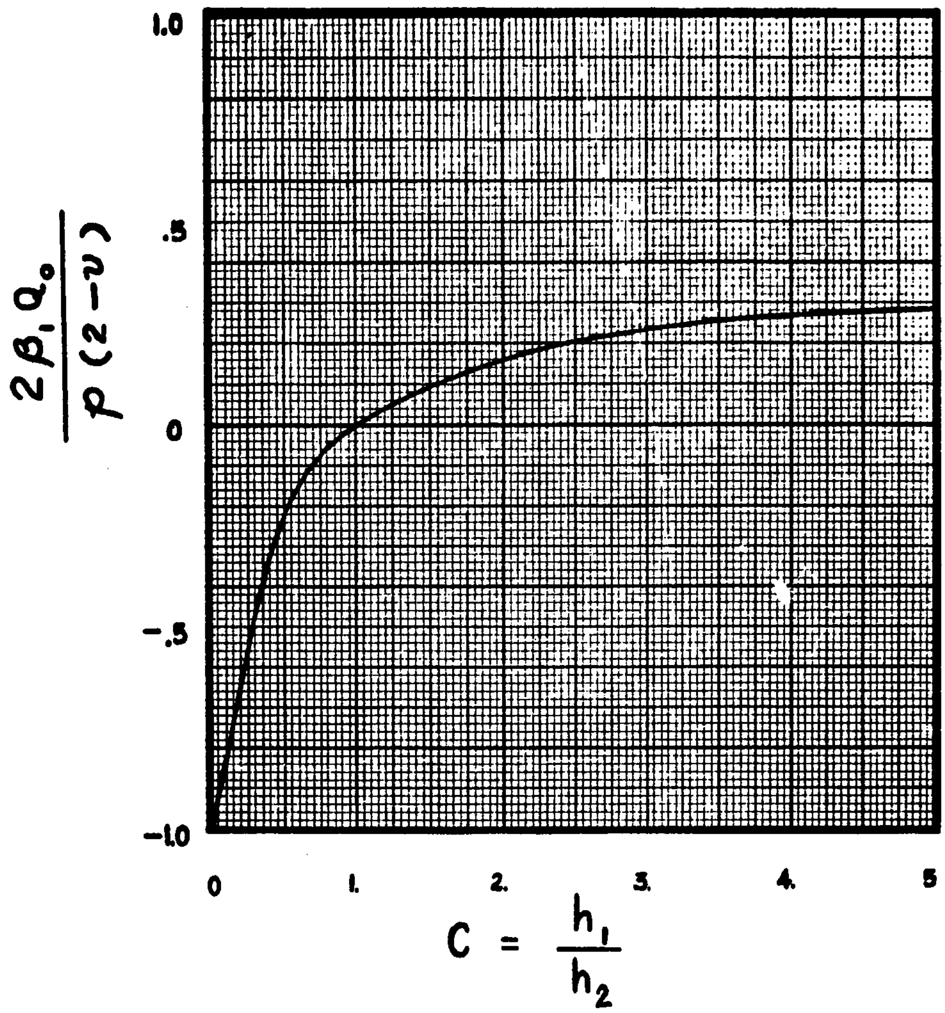


Fig. 5.2 Variation of Juncture Shearing Forces with Thickness Ratio for Two Long Cylindrical Shells of Unequal Thickness

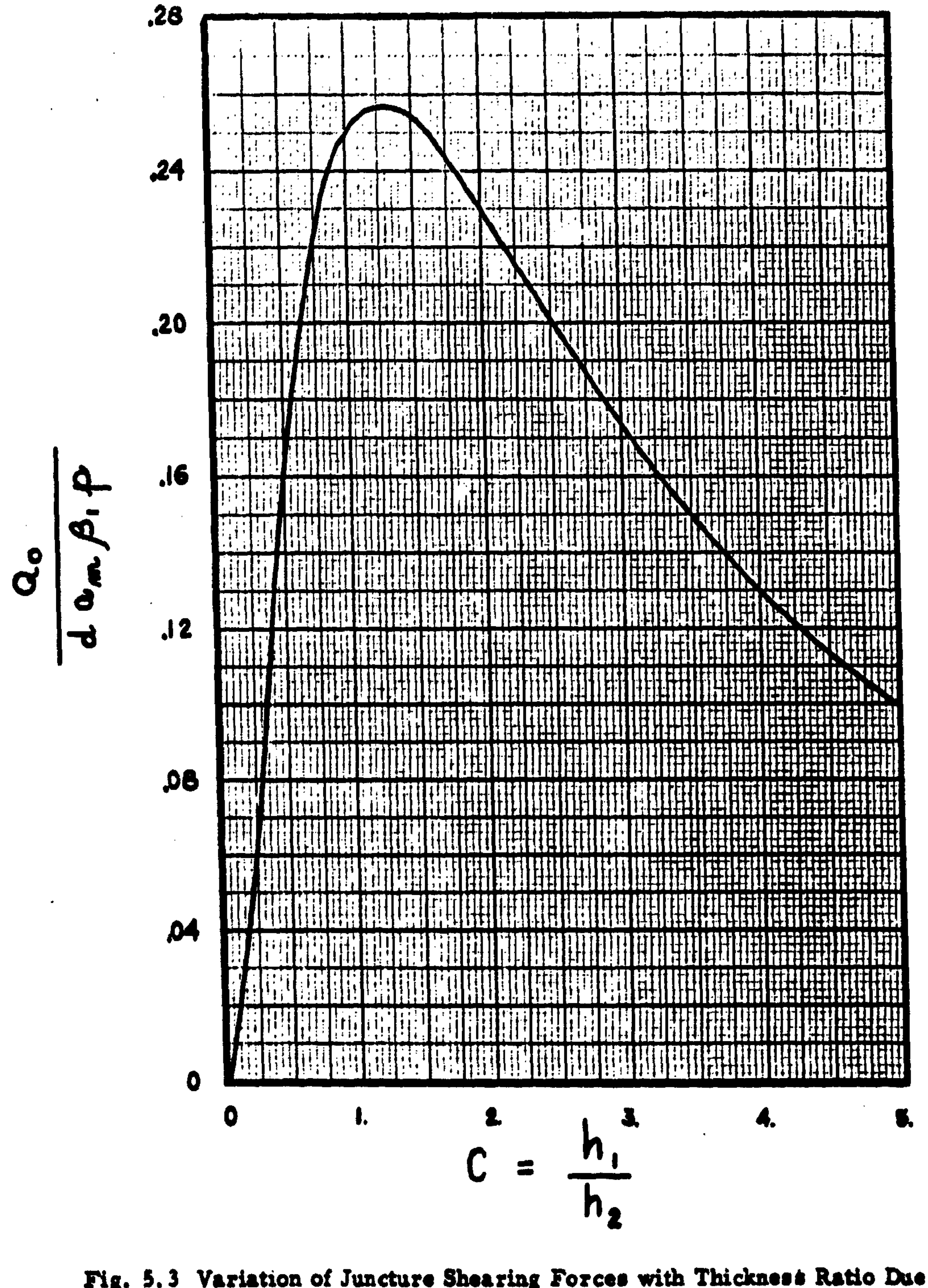


Fig. 5.3 Variation of Juncture Shearing Forces with Thickness Ratio Due to Mismatch of Middle Surfaces Only of Two Long Cylinders of Unequal Thickness (to be used in conjunction with Fig. 5.1)

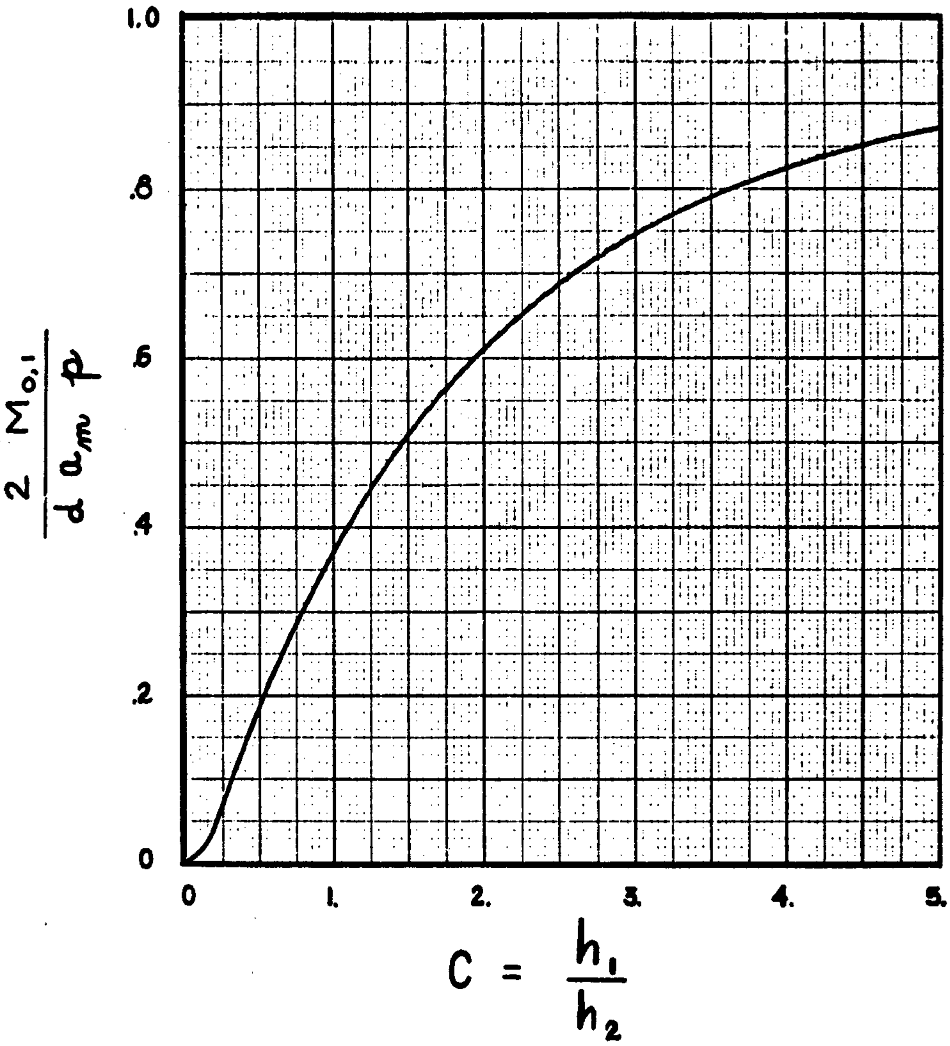


Fig. 5.4 Variation of Juncture Bending Moments with Thickness Ratio Due to Mismatch of Middle Surfaces Only of Two Long Cylinders of Unequal Thickness (to be used in conjunction with Fig. 5.2)

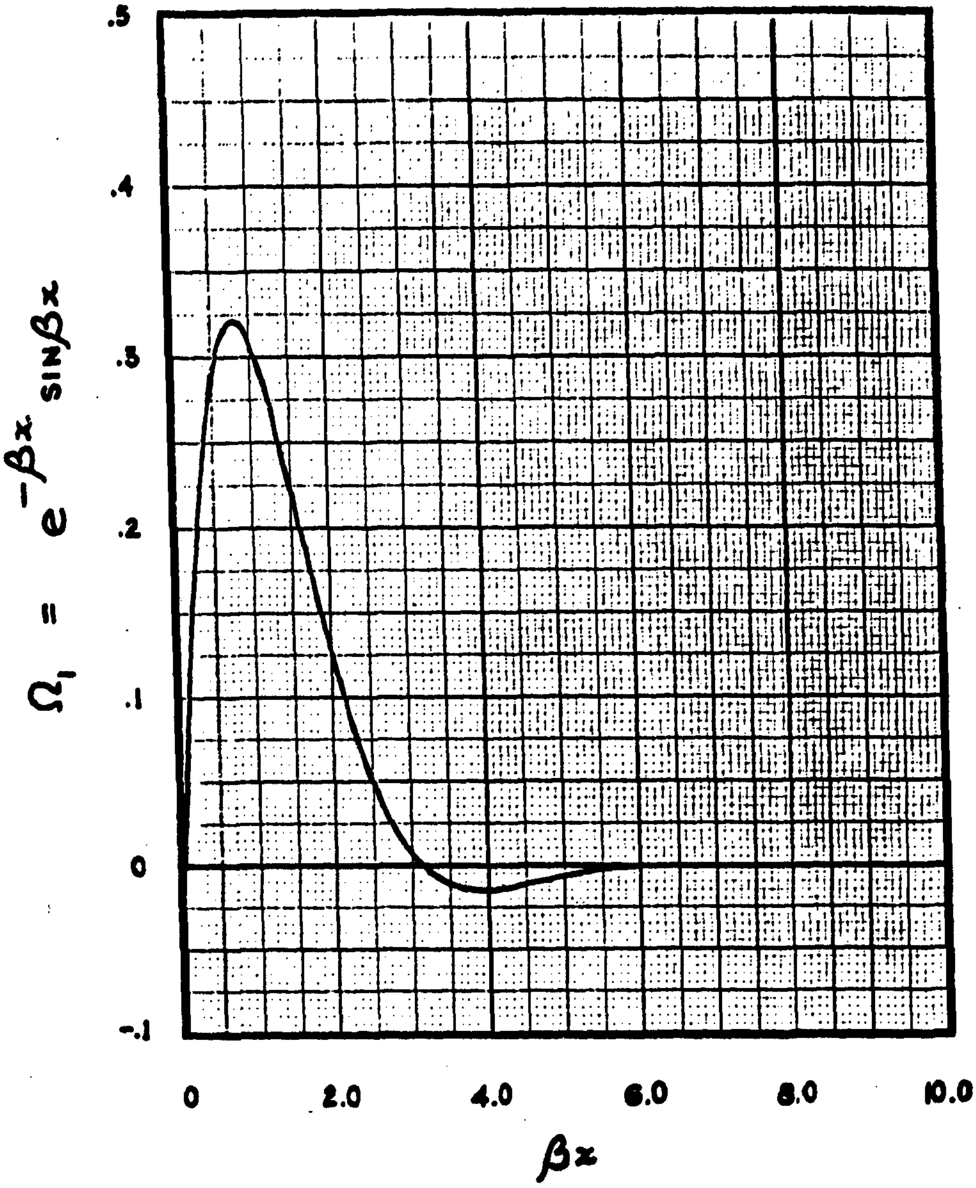


Fig. 5.5 Variation of Ω_1 with βx

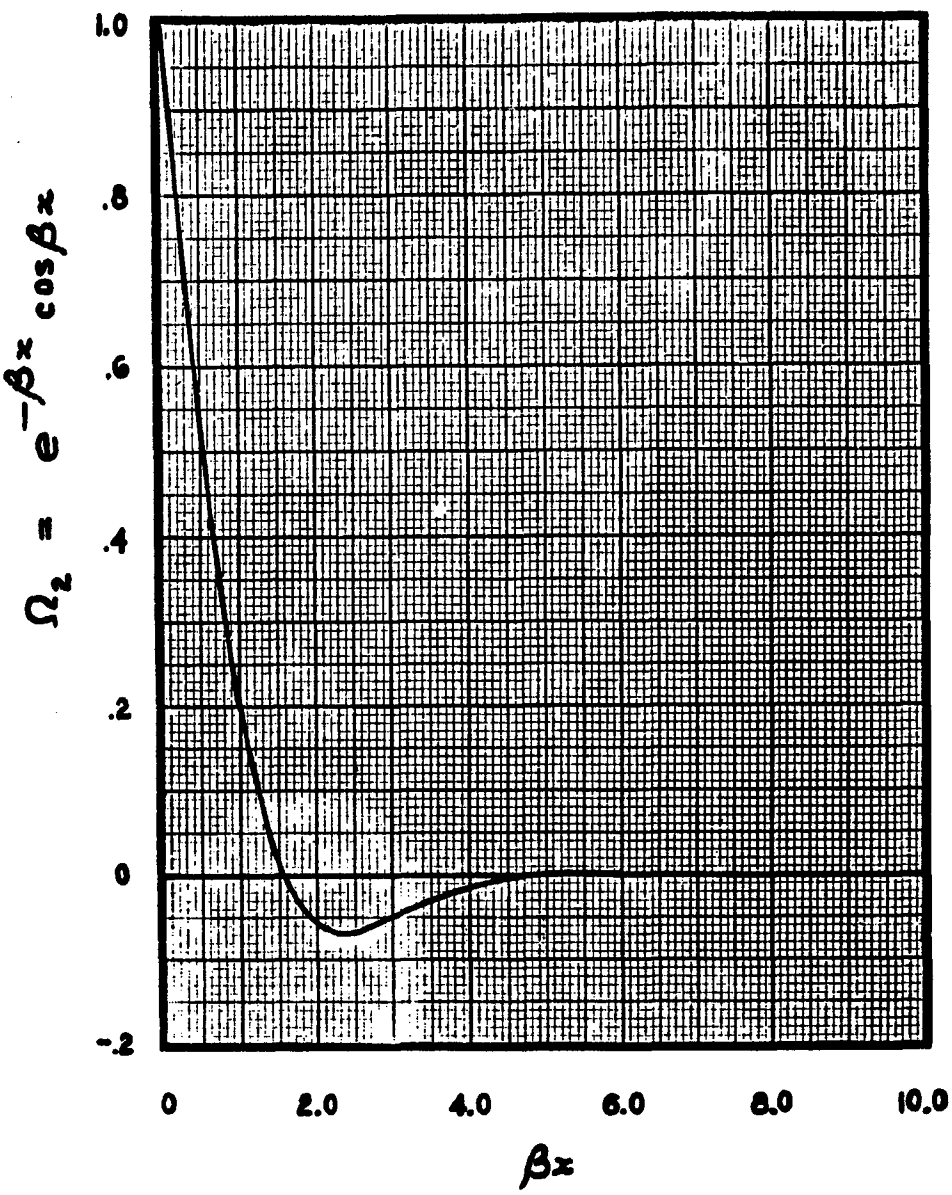


Fig. 5.6 Variation of Ω_2 with βx

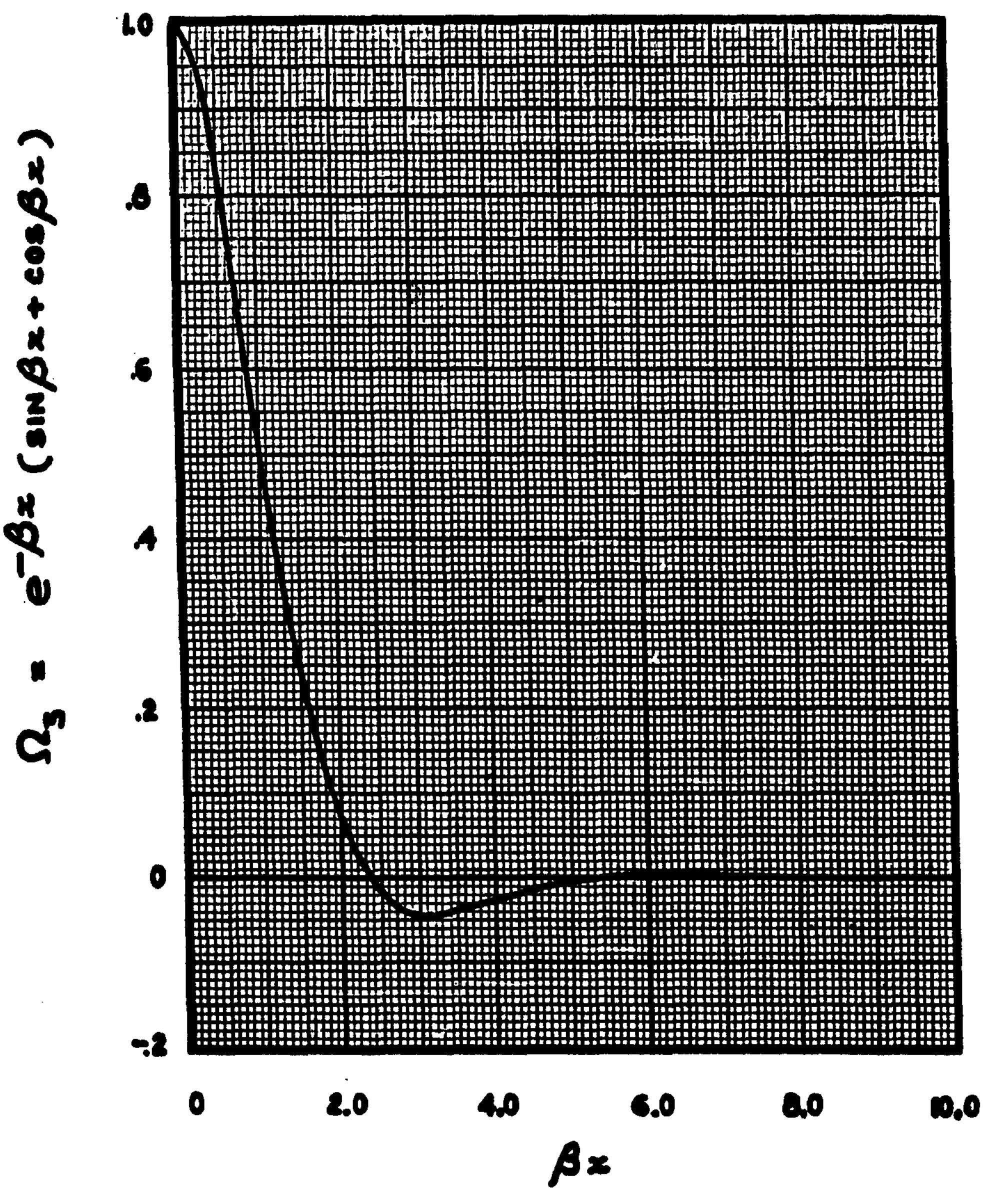


Fig. 5.7 Variation of Ω_3 with βx

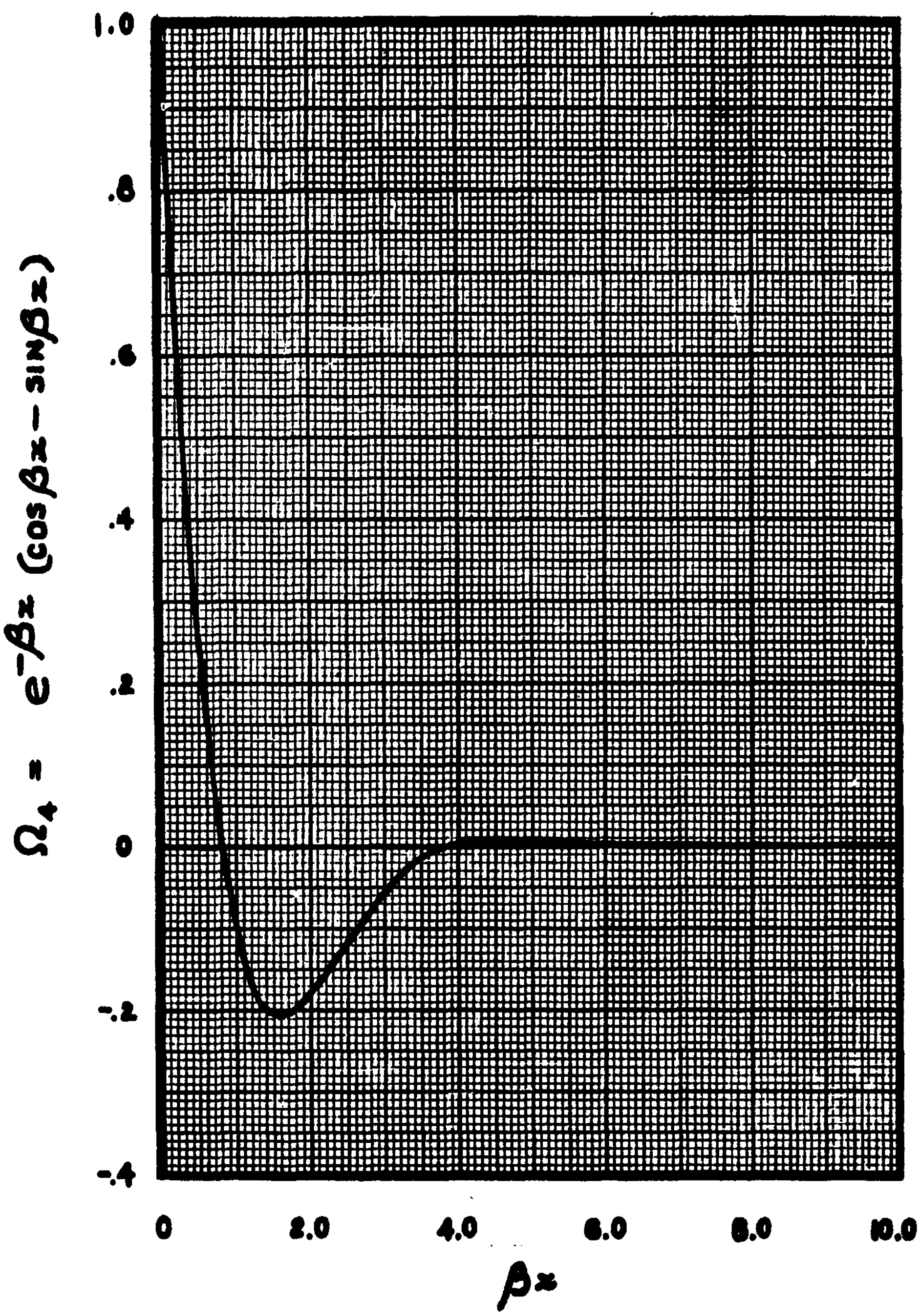


Fig. 5.8 Variation of Ω_4 with βx

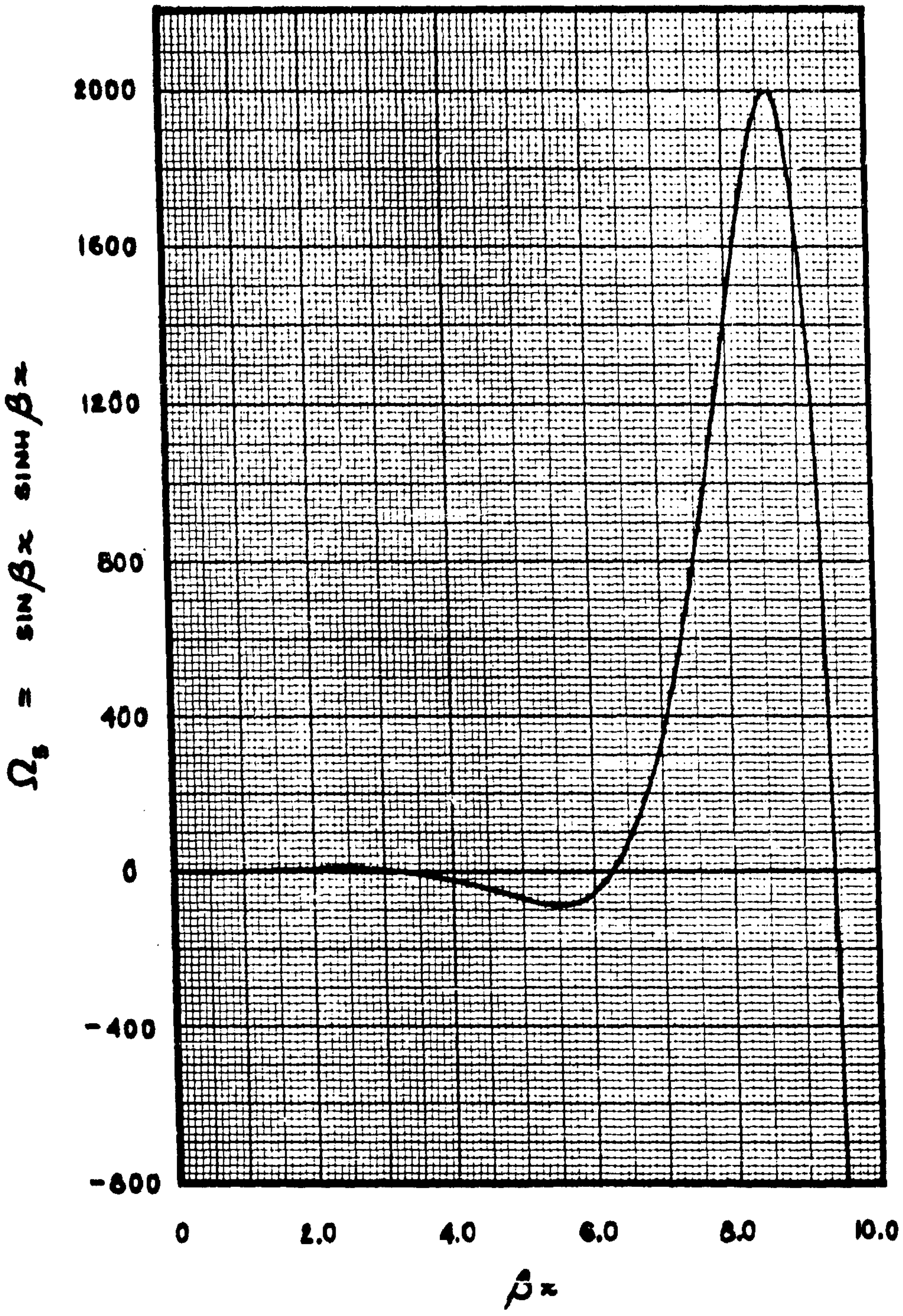


Fig. 5.9 Variation of Ω_5 with βx

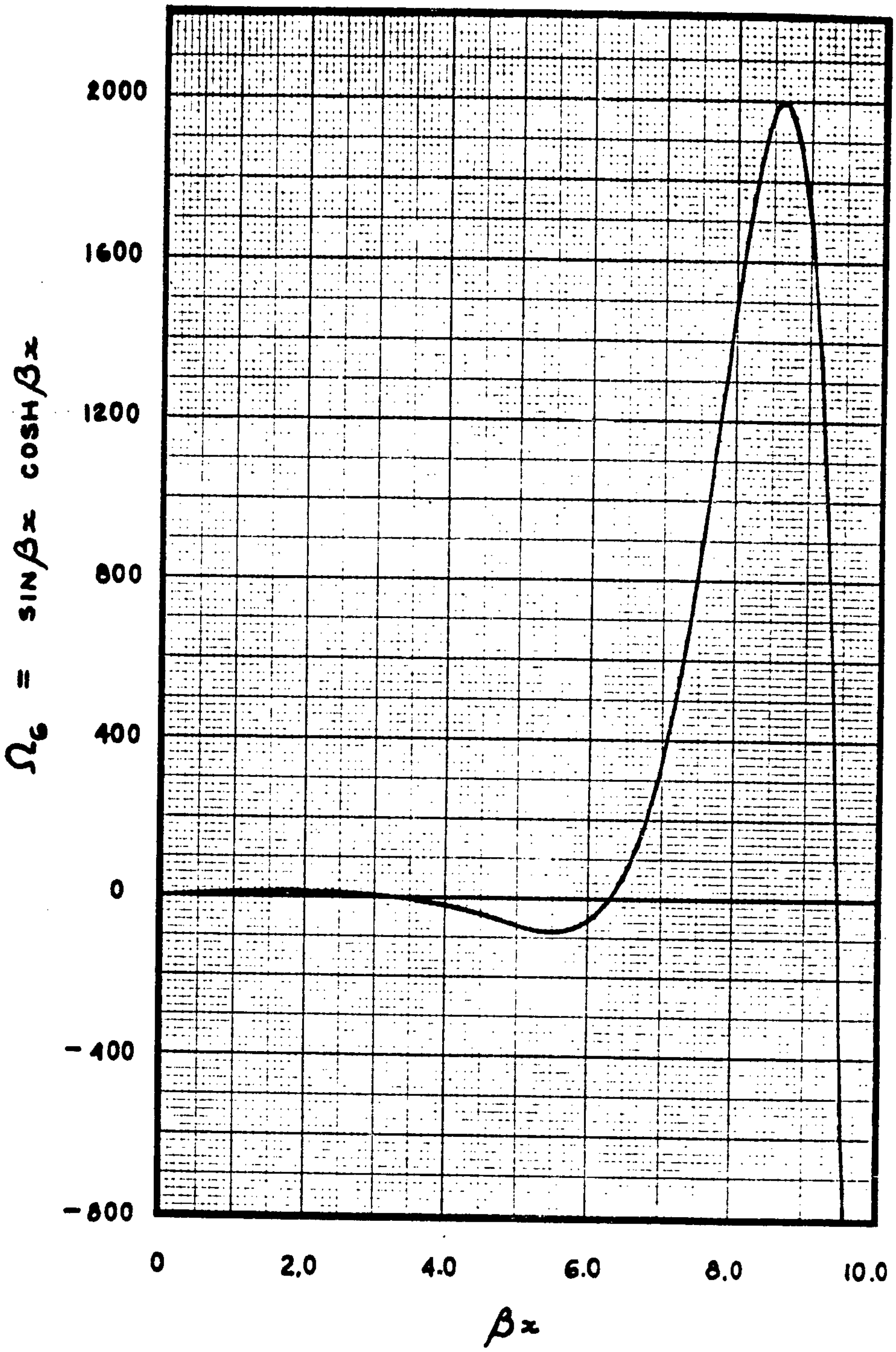


Fig. 5.10 Variation of Ω_6 with βx

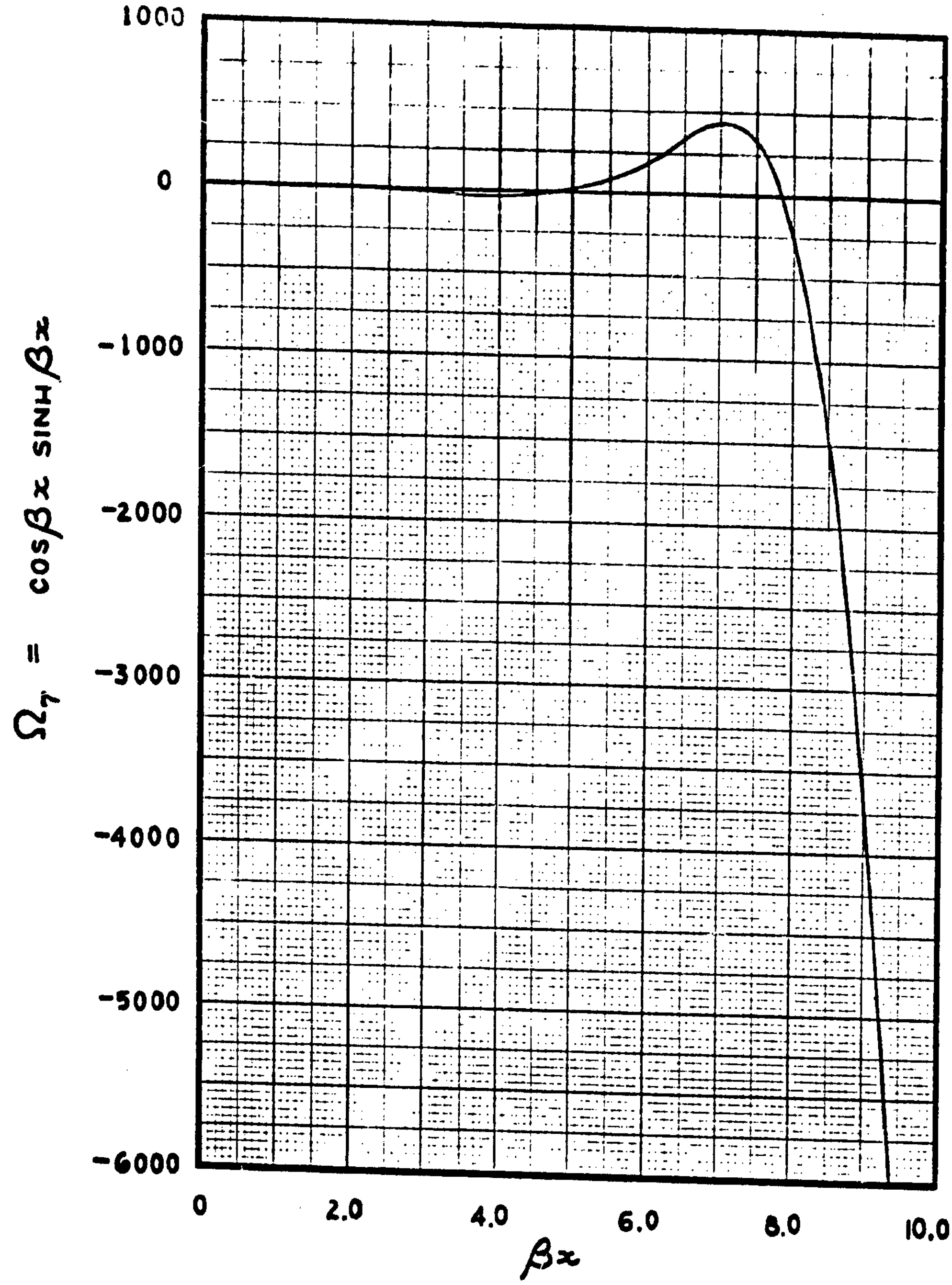


Fig. 5.11 Variation of Ω_7 with βx

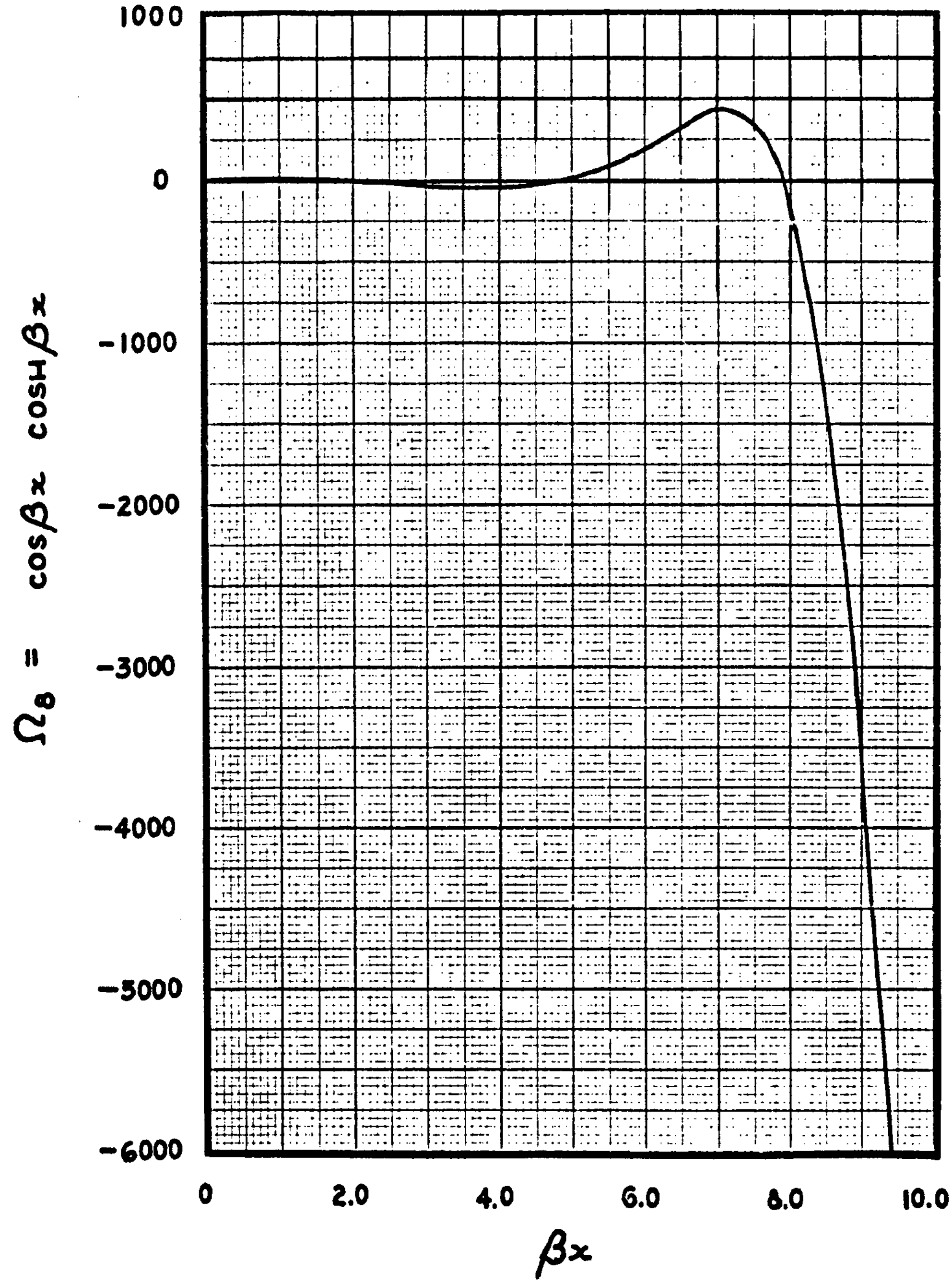


Fig. 5.12 Variation of Ω_8 with βx

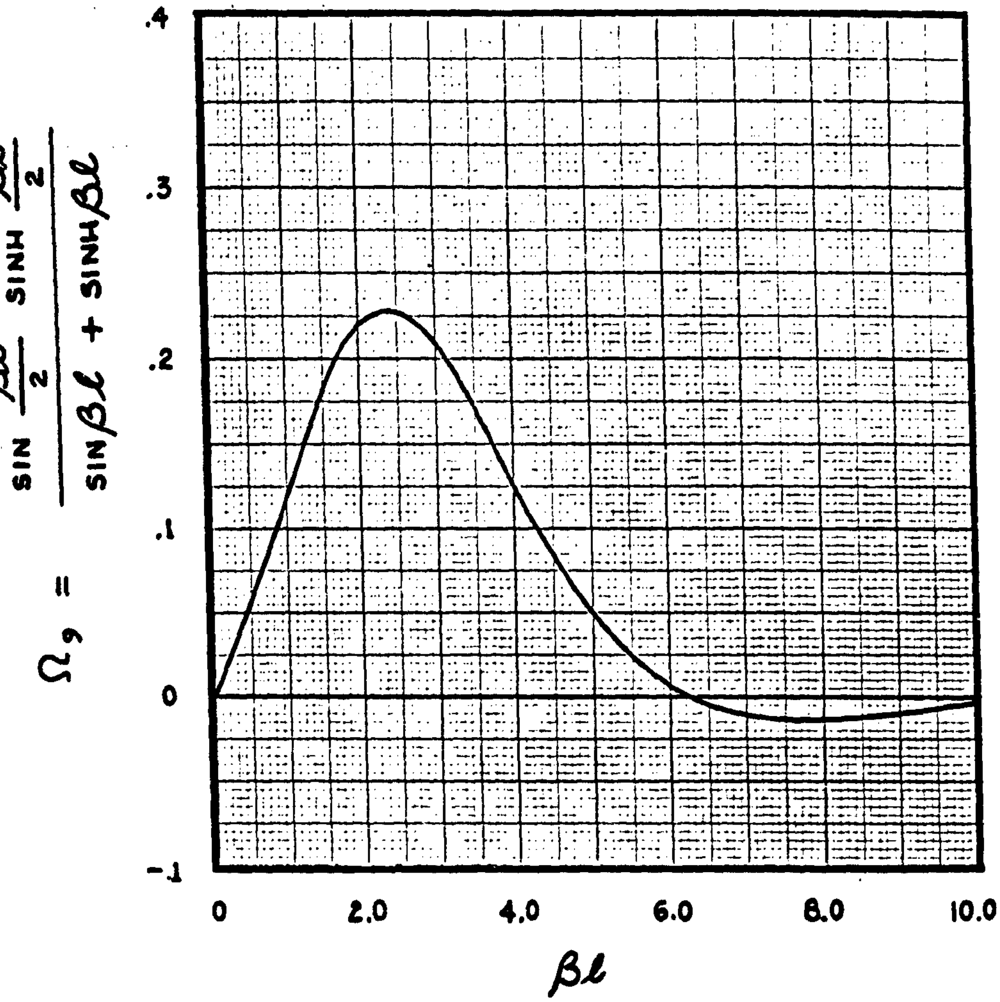


Fig. 5.13 Variation of Ω_9 with $\beta\ell$

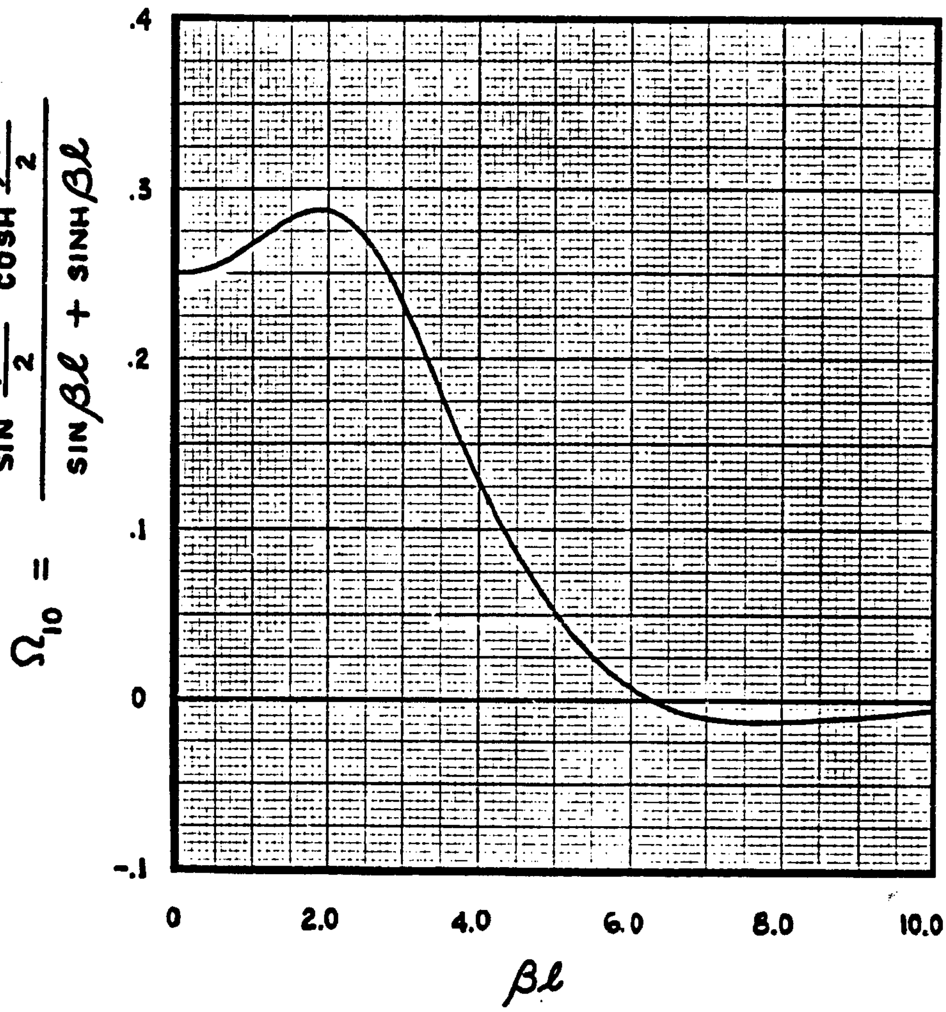


Fig. 5.14 Variation of Ω_{10} with βl

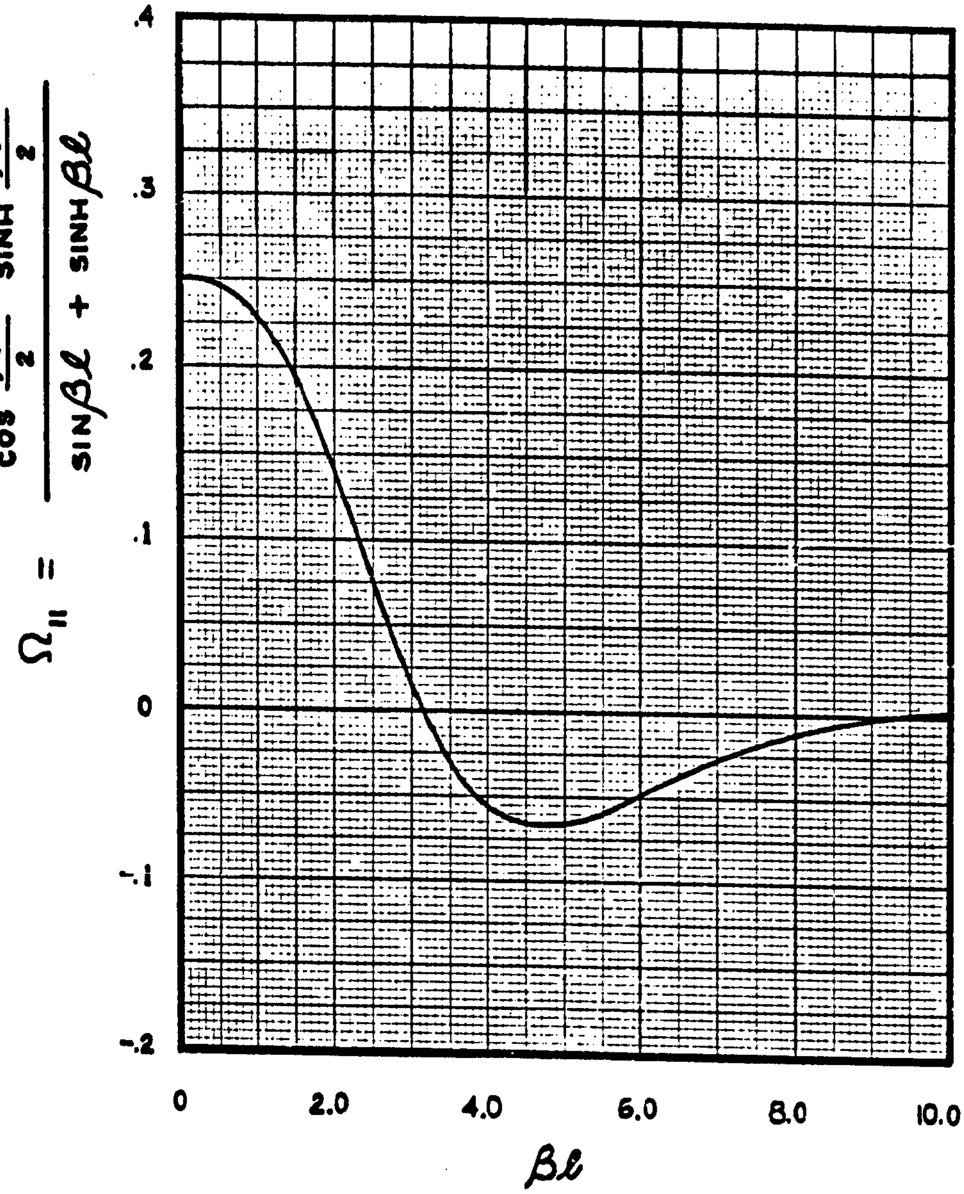


Fig. 5.15 Variation of Ω_{11} with $\beta\ell$

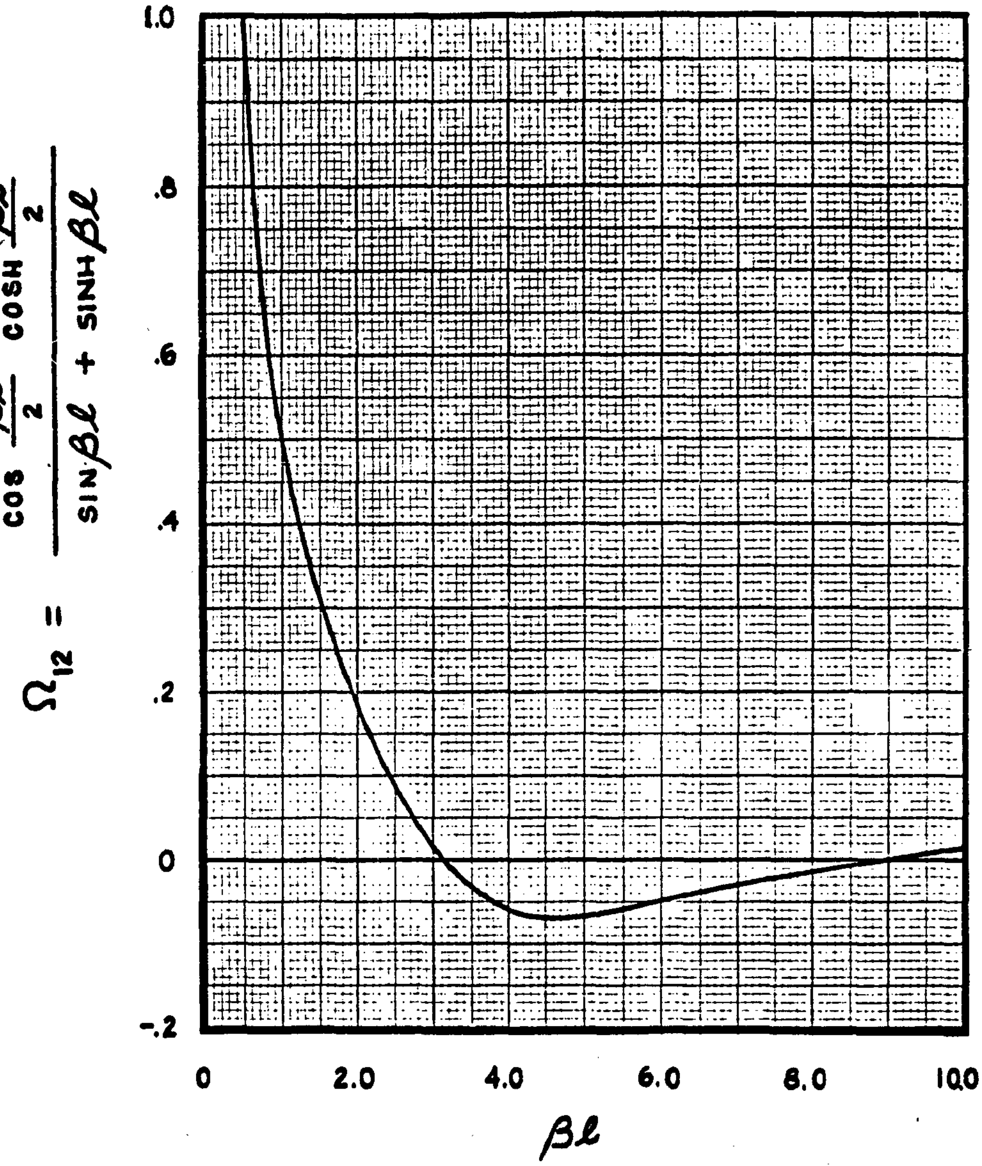


Fig. 5.16 Variation of Ω_{12} with $\beta\ell$

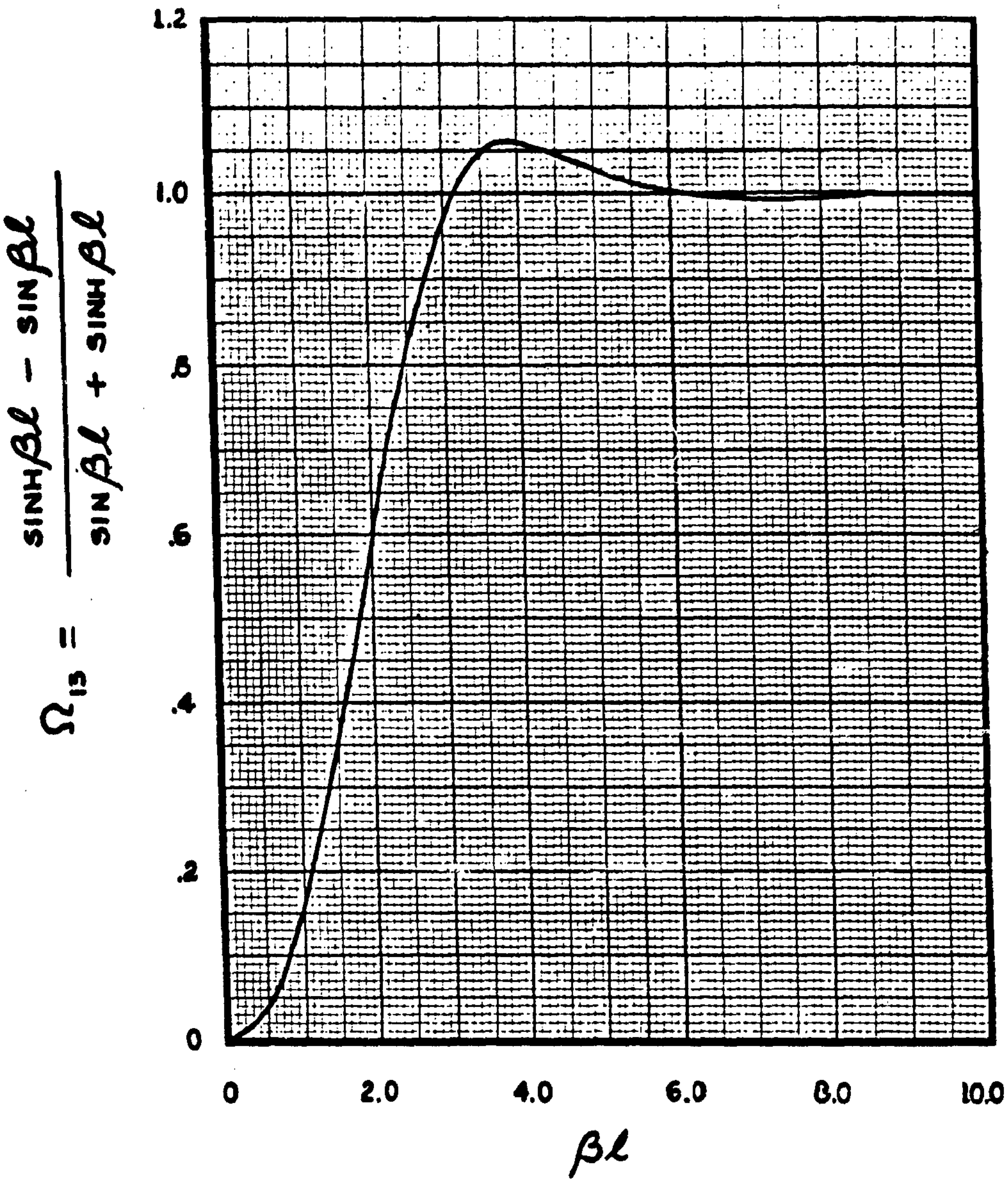


Fig. 5.17 Variation of Ω_{13} with $\beta\ell$

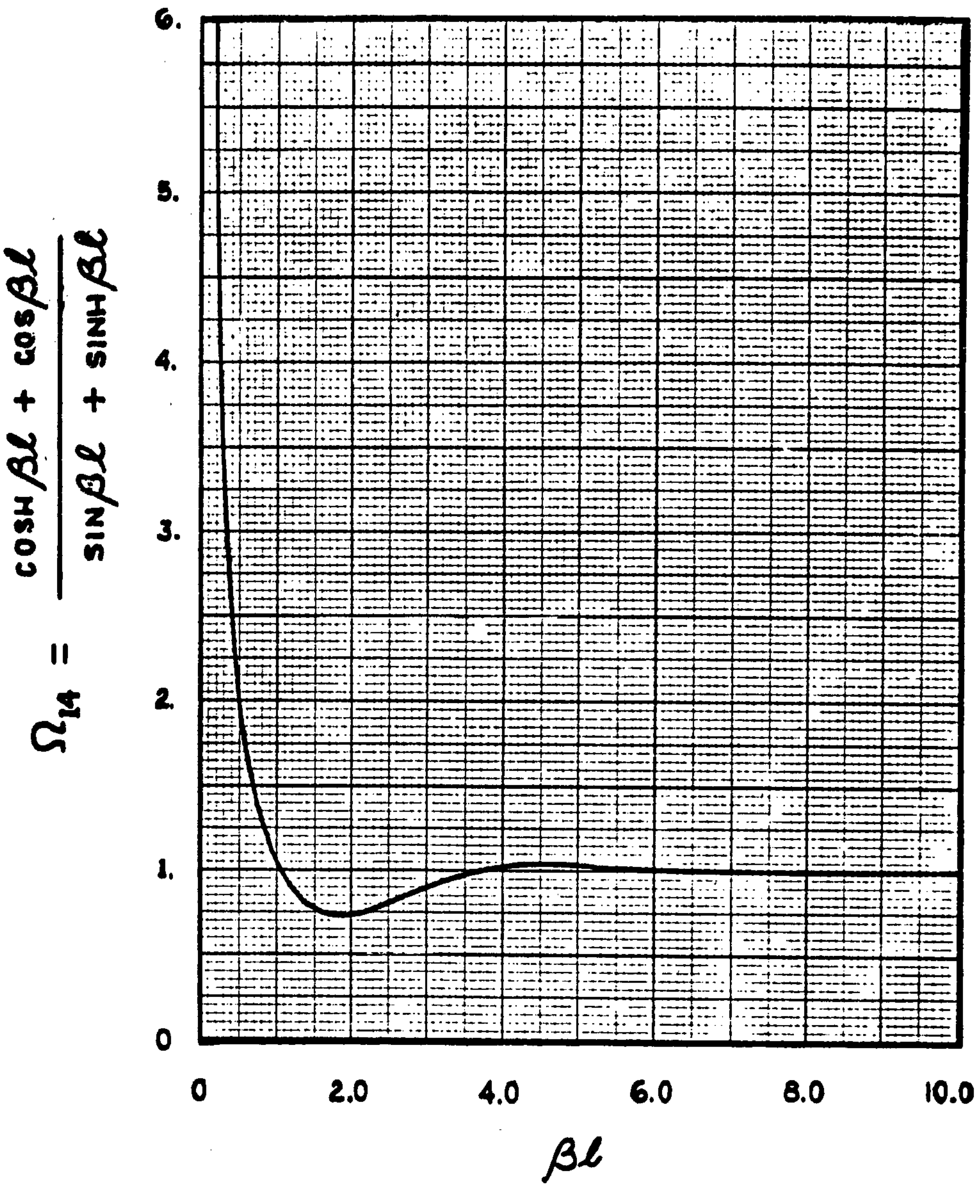


Fig. 5.18 Variation of Ω_{14} with βl

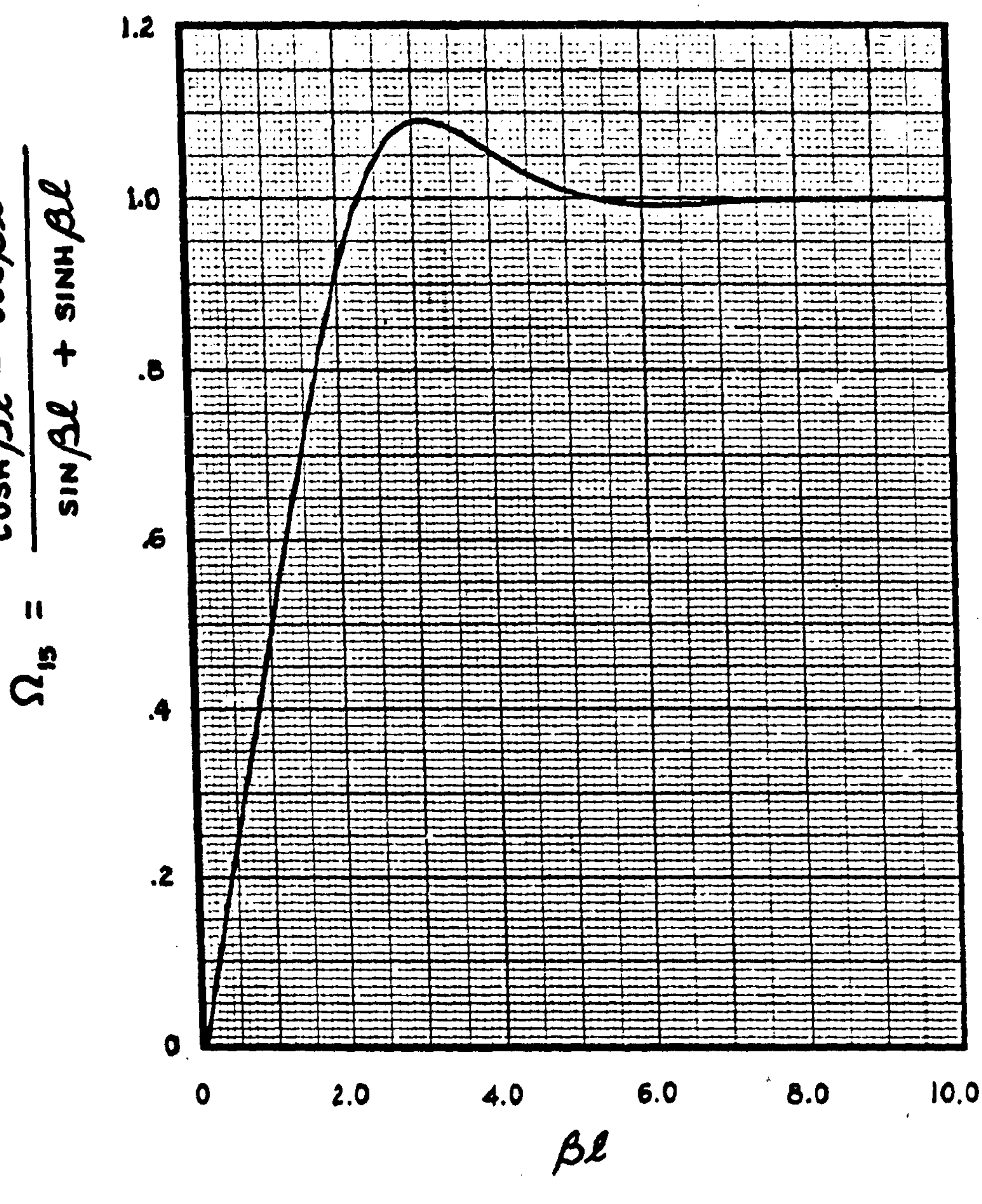


Fig. 5.19 Variation of Ω_{15} with $\beta\ell$

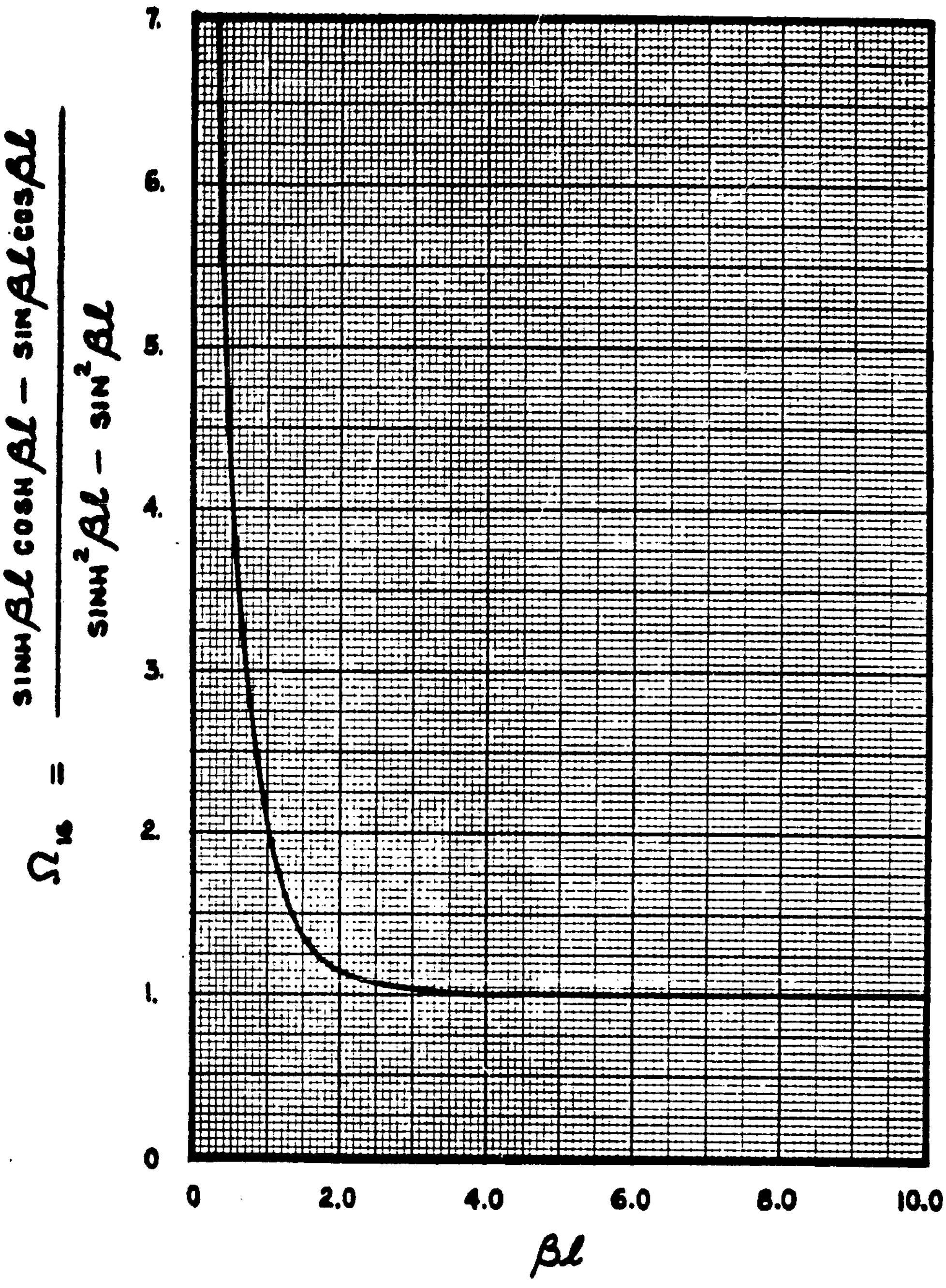


Fig. 5.20 Variation of Ω_{16} with $\beta\ell$

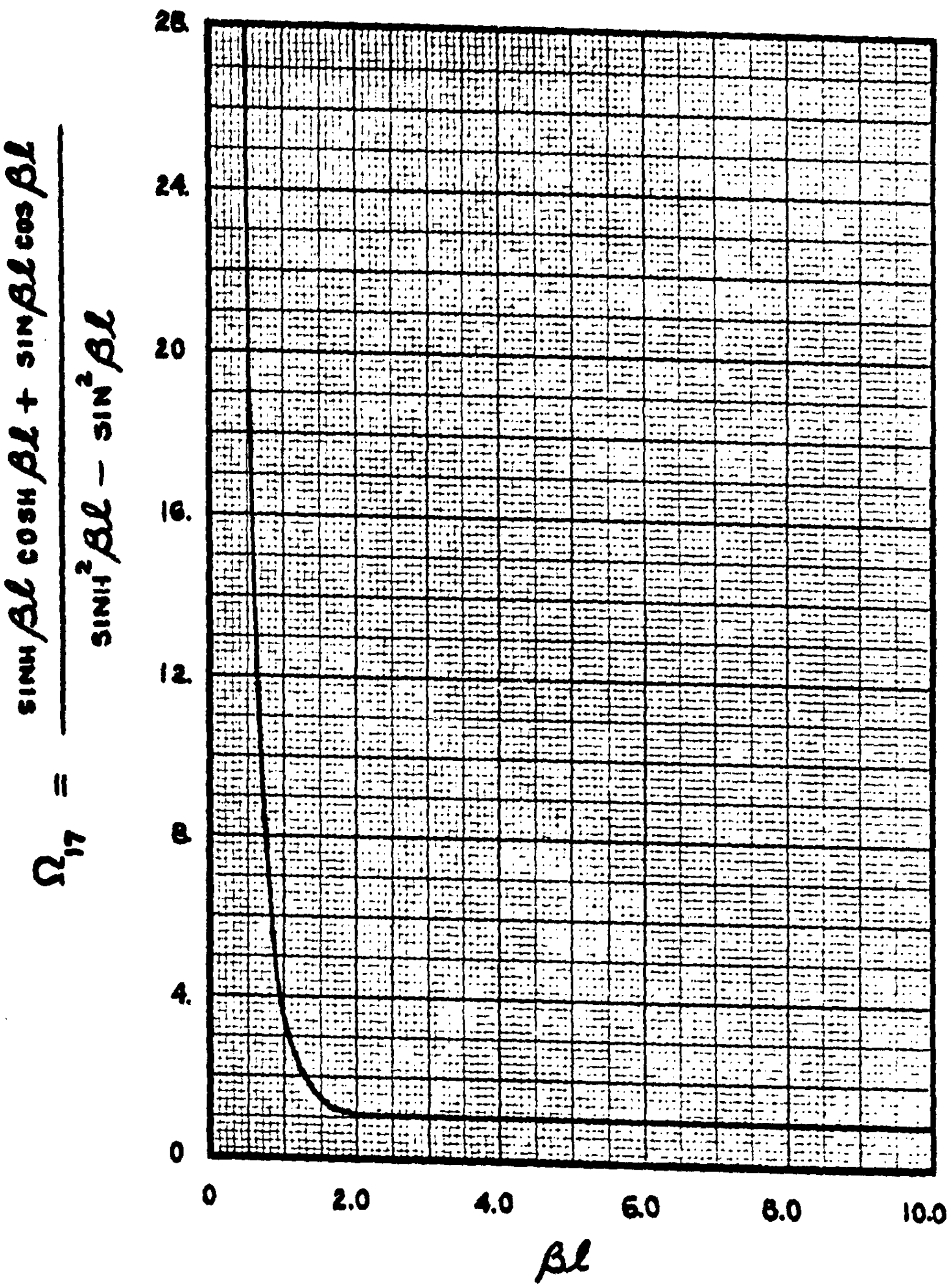


Fig. 5.21 Variation of Ω_{17} with $\beta\ell$

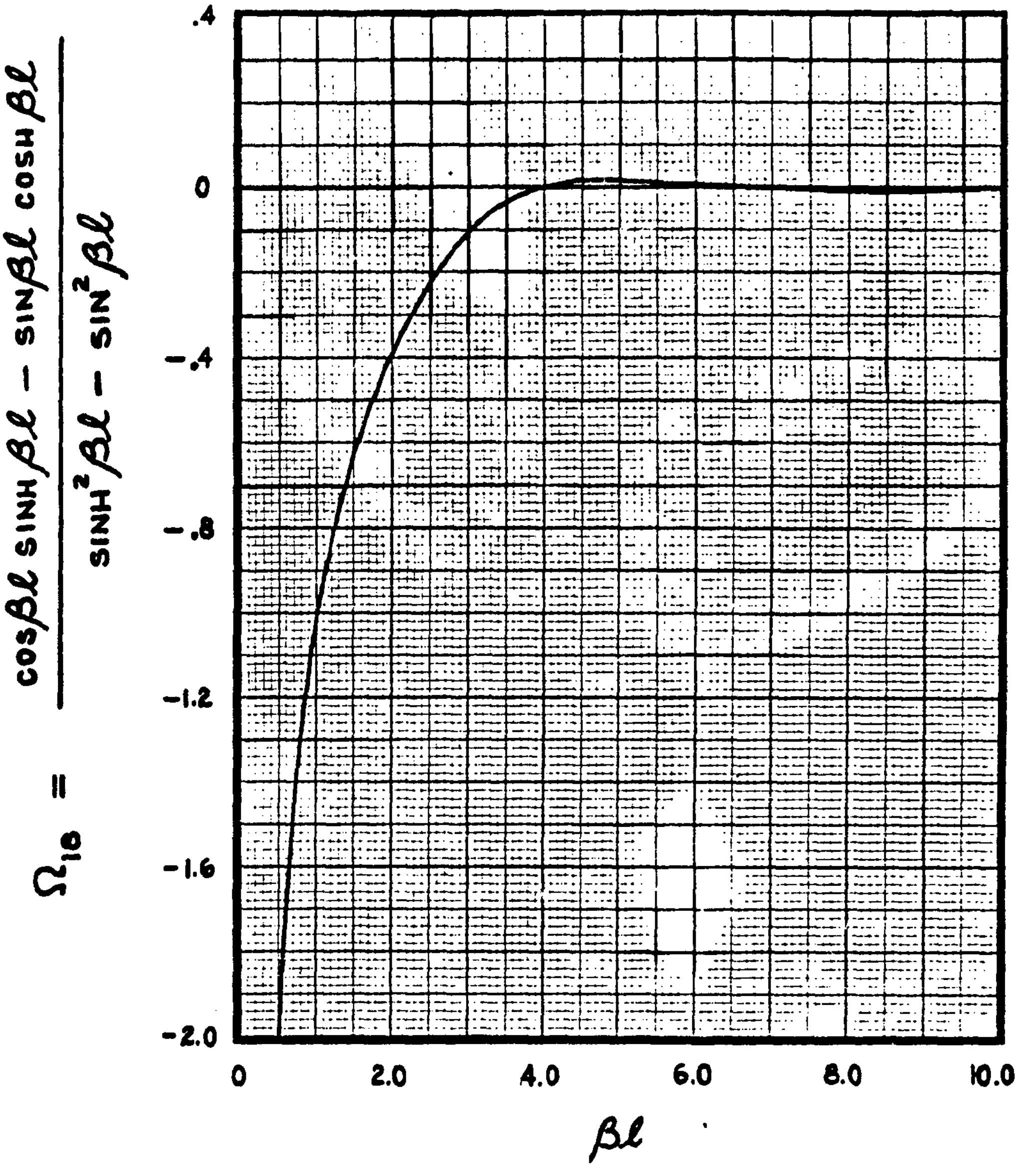


Fig. 5.22 Variation of Ω_{18} with βl

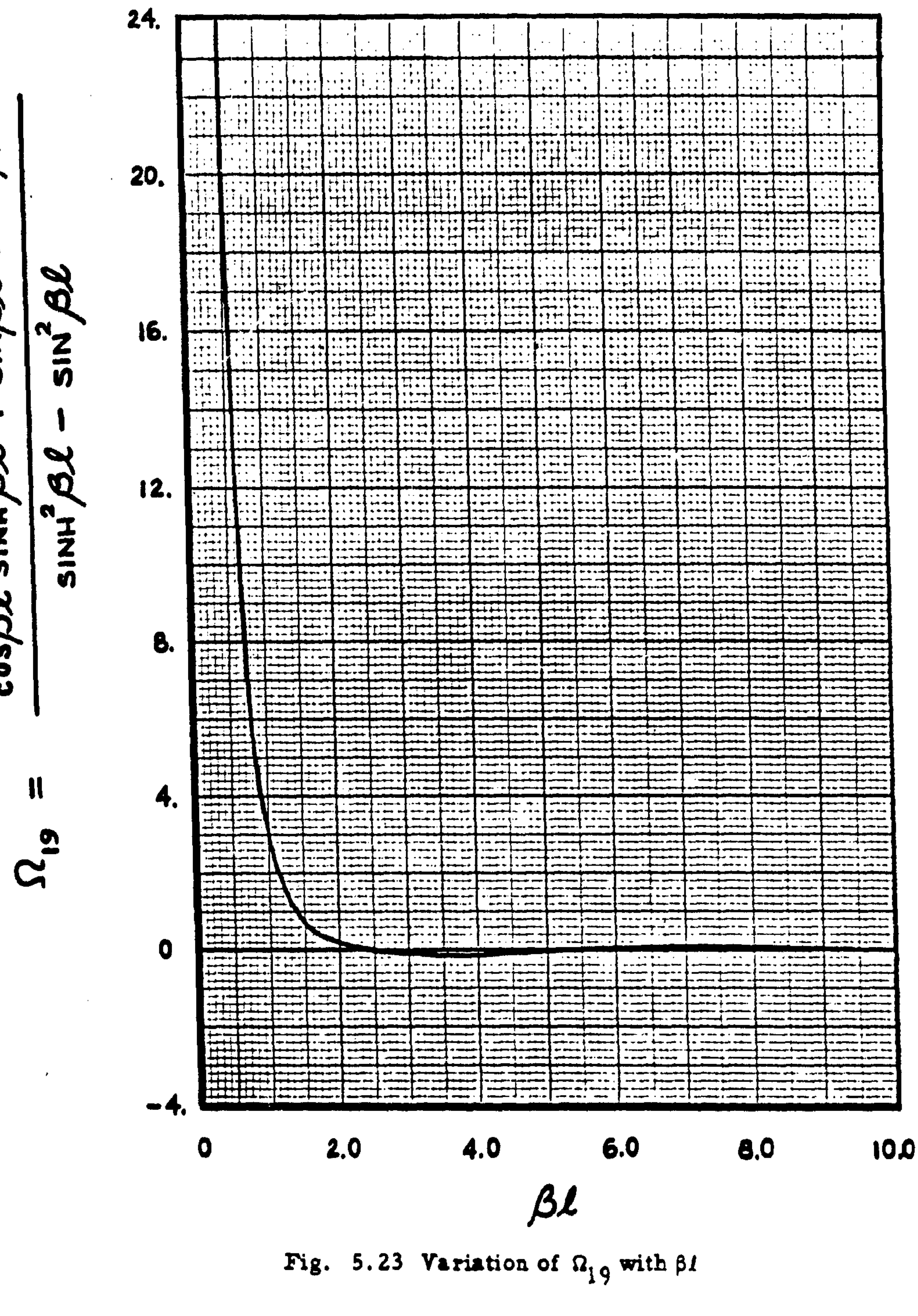


Fig. 5.23 Variation of Ω_{19} with βl

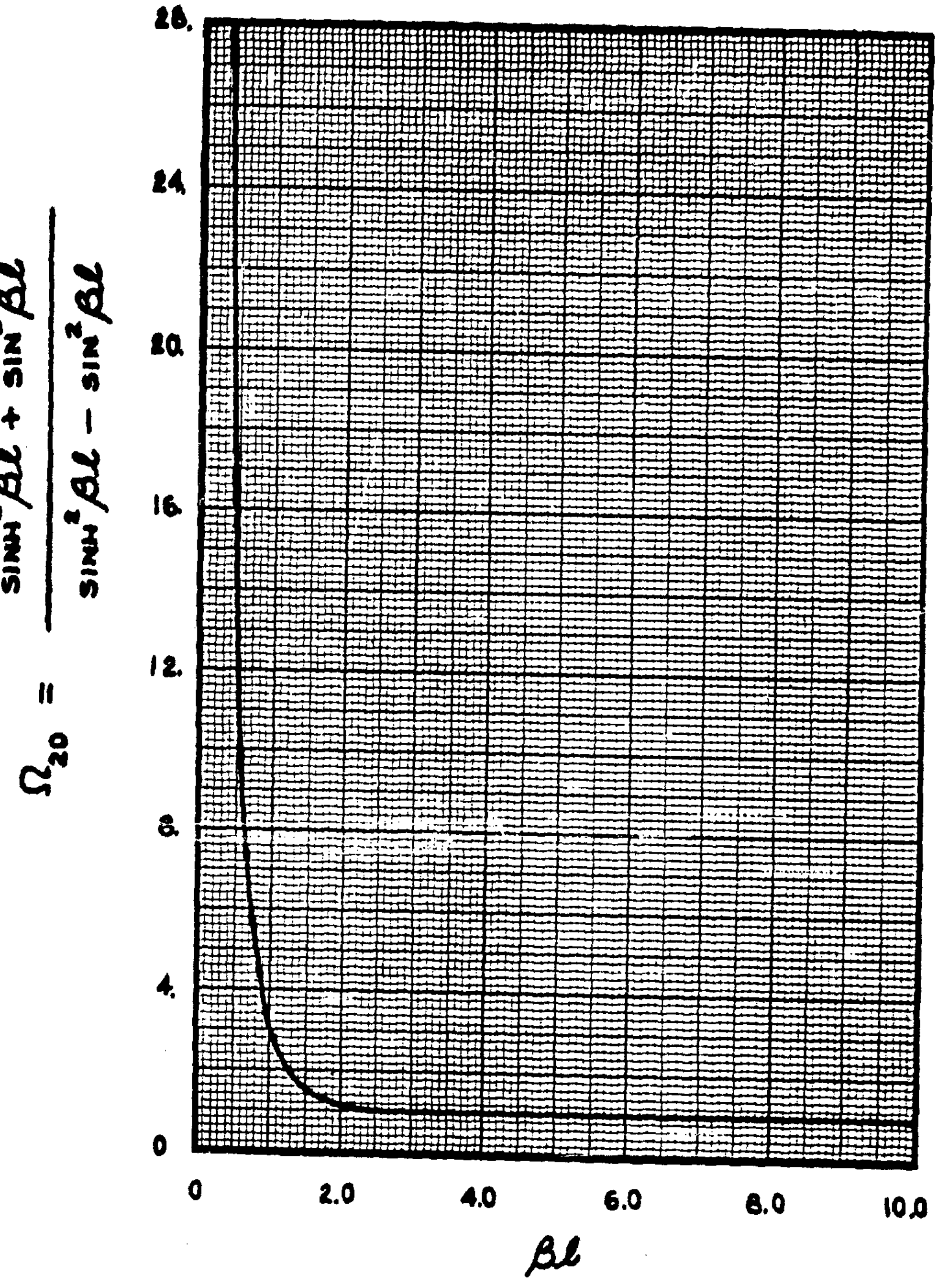


Fig. 5.24 Variation of Ω_{20} with βl

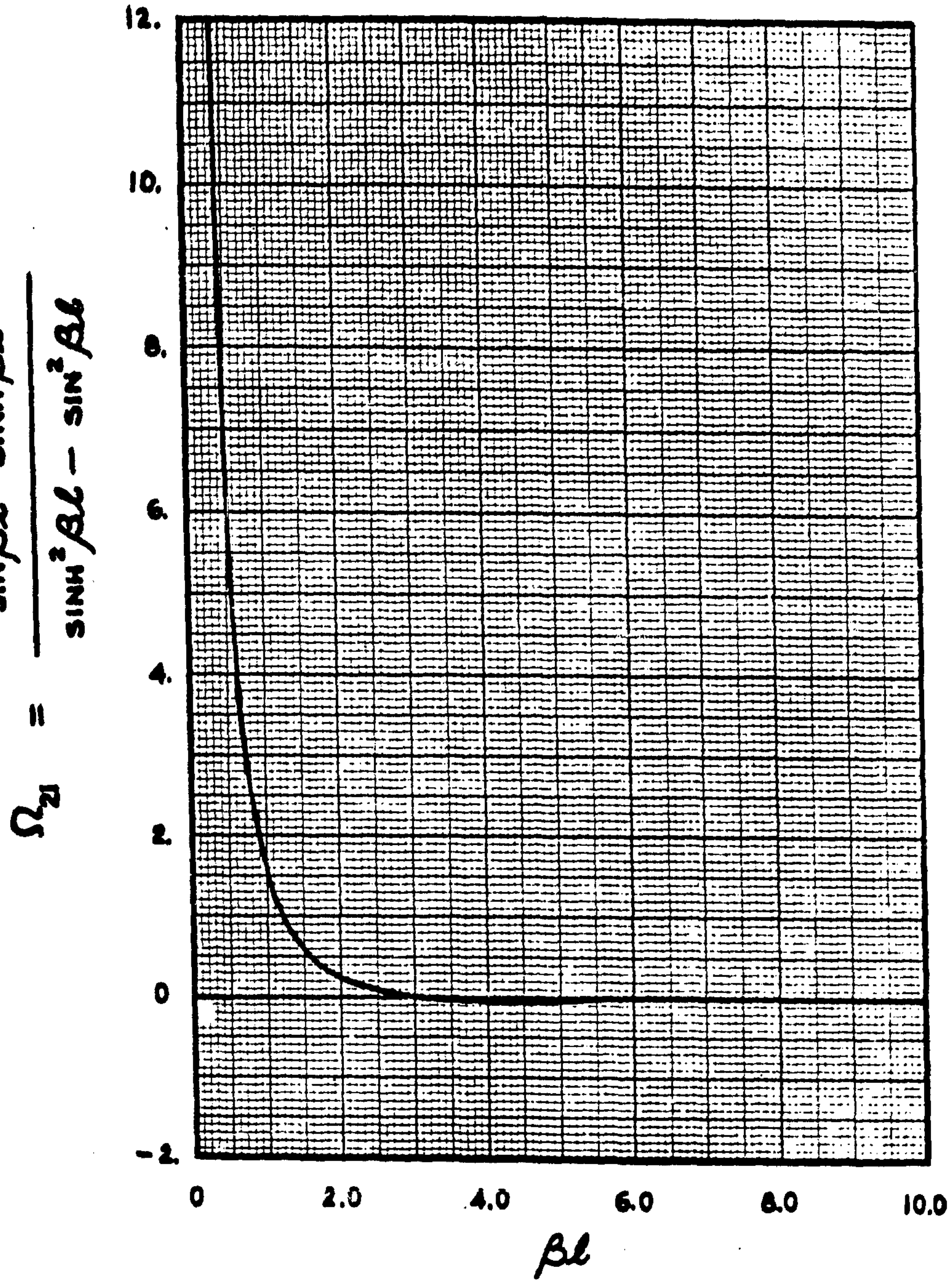


Fig. 5.25 Variation of Ω_{21} with $\beta\ell$

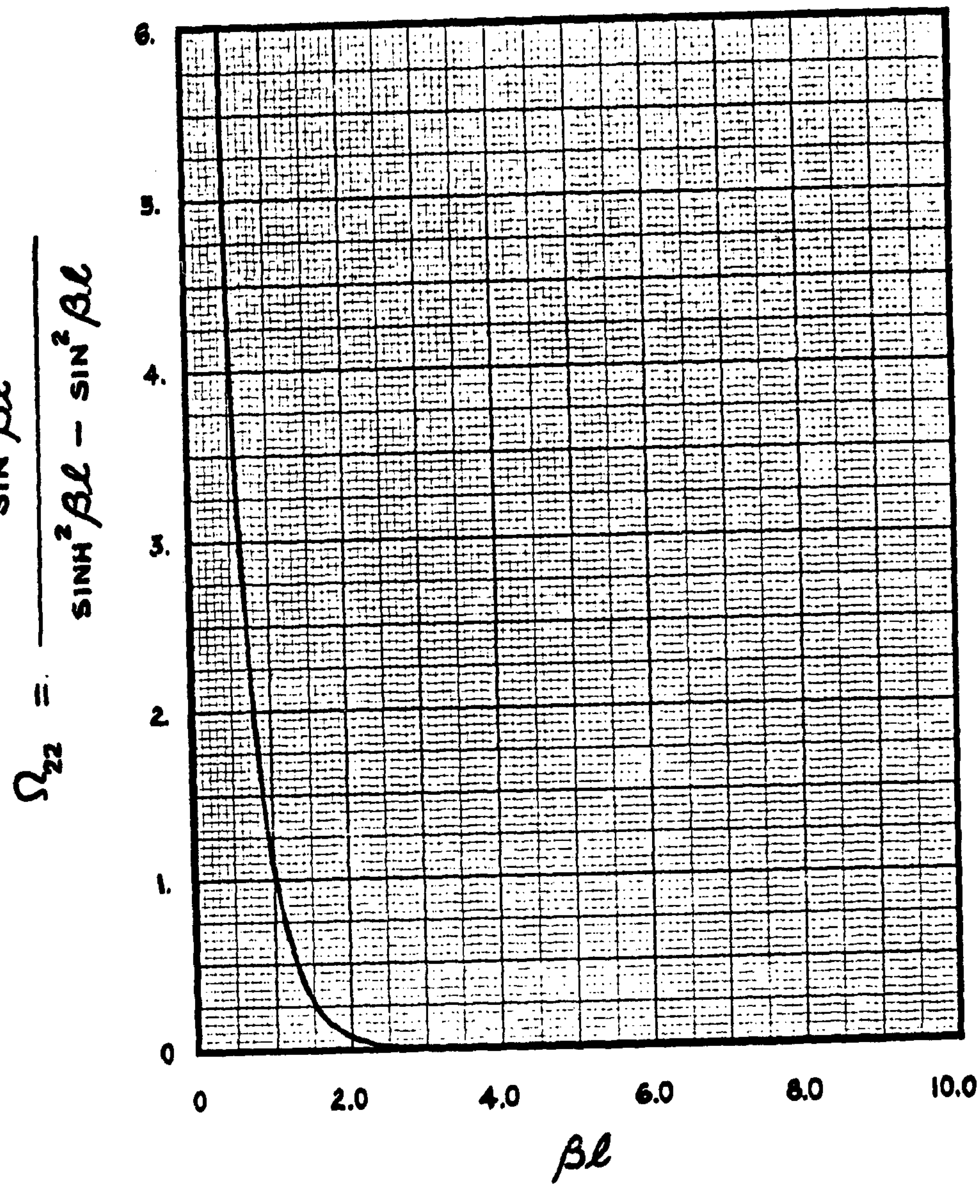


Fig. 5.26 Variation of Ω_{22} with $\beta\ell$

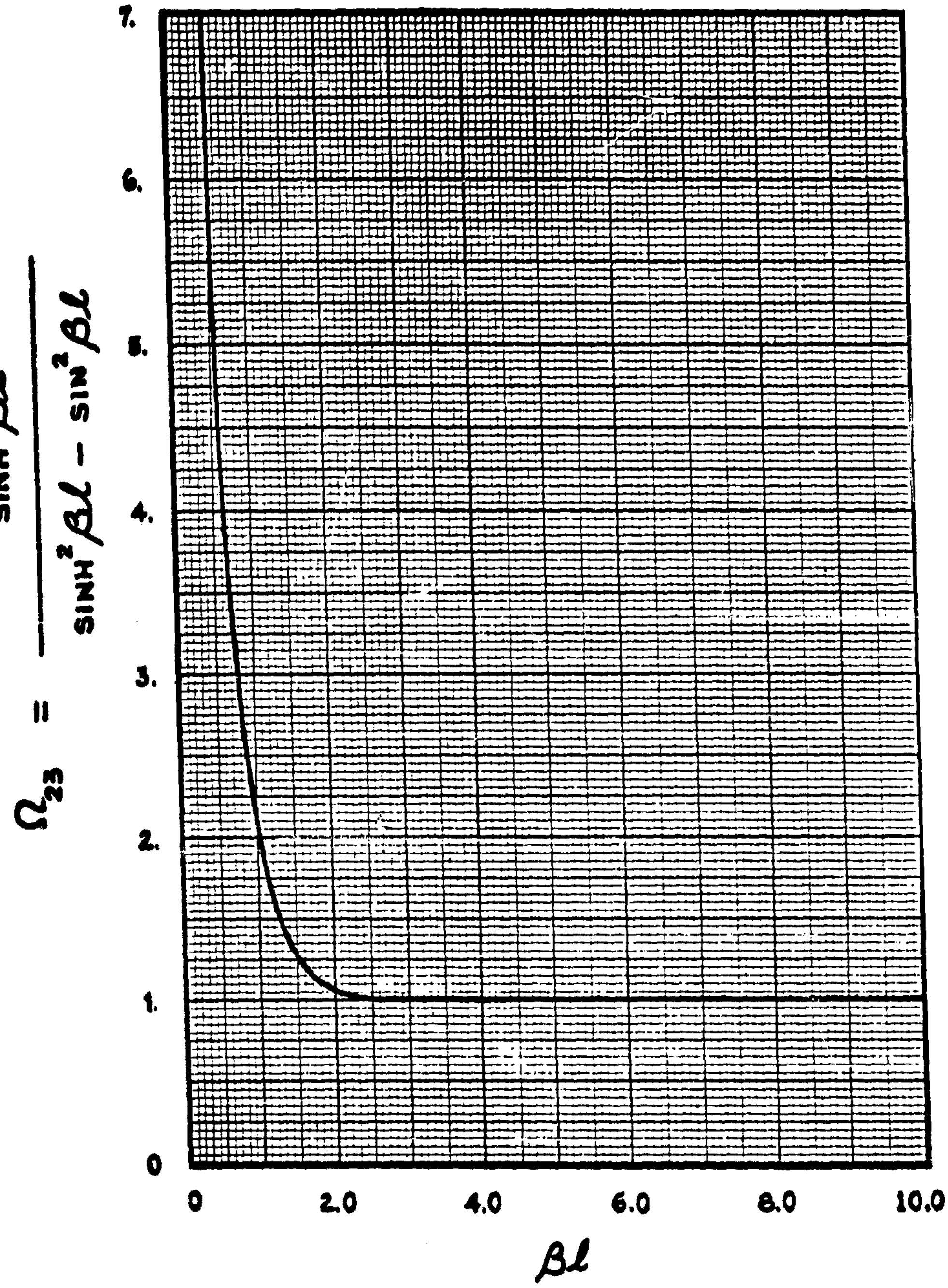


Fig. 5.27 Variation of Ω_{23} with βl

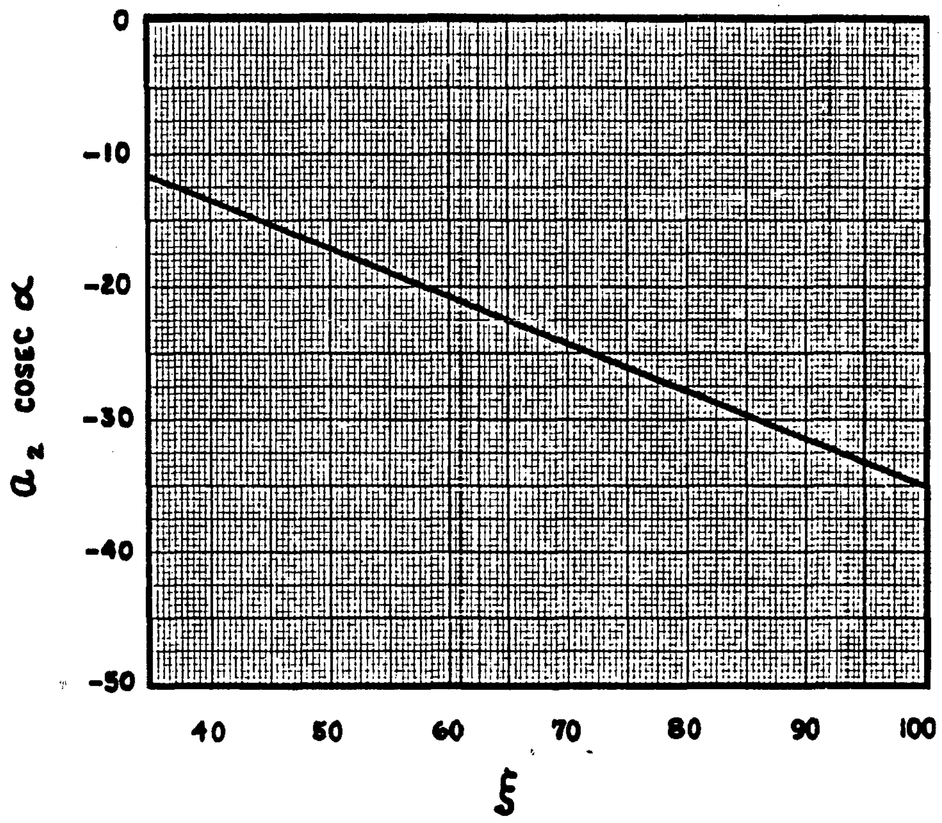


Fig. 5.28. Variation of a_2 cosec α with ξ

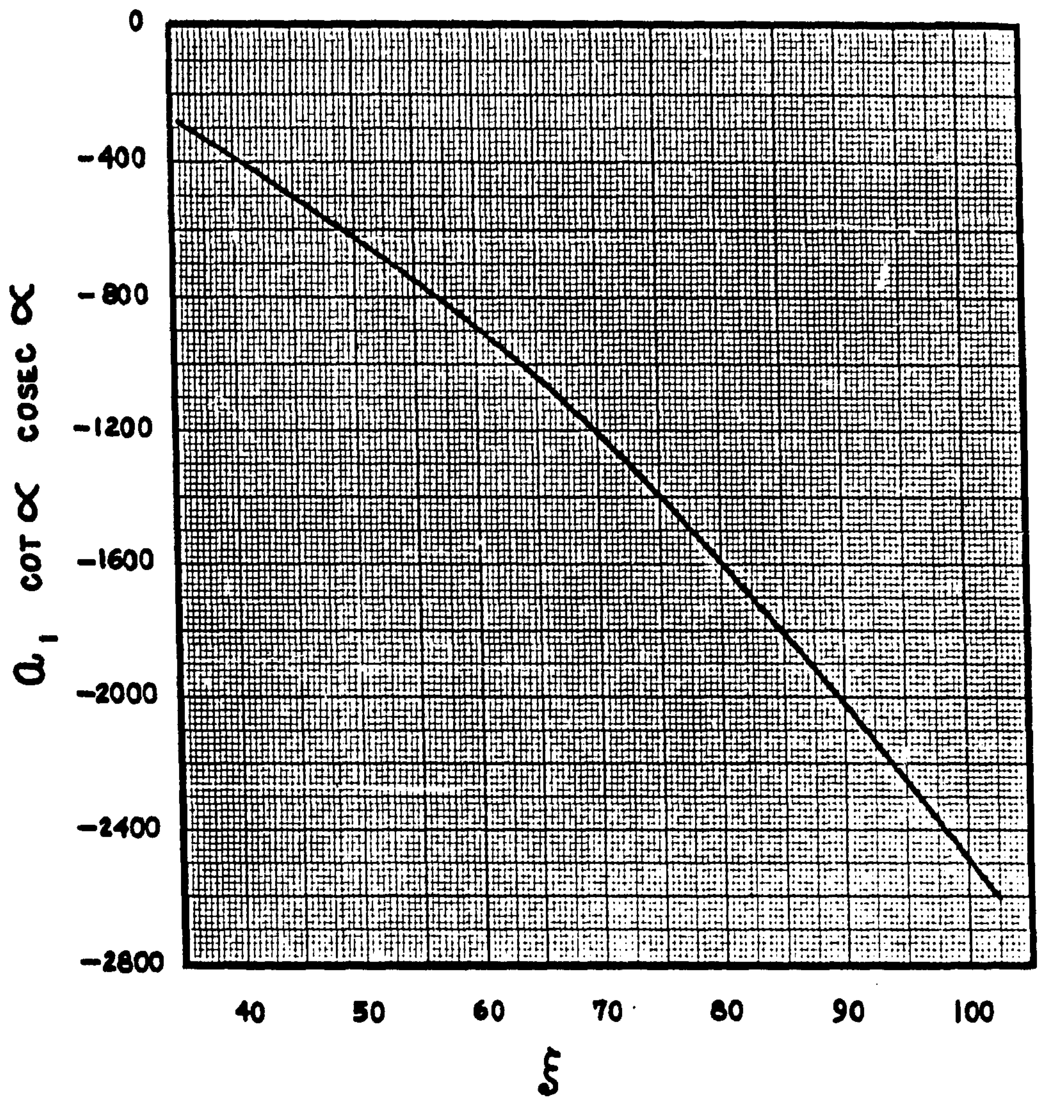


Fig. 5.29 Variation of a_1 cot α cosec α with ξ

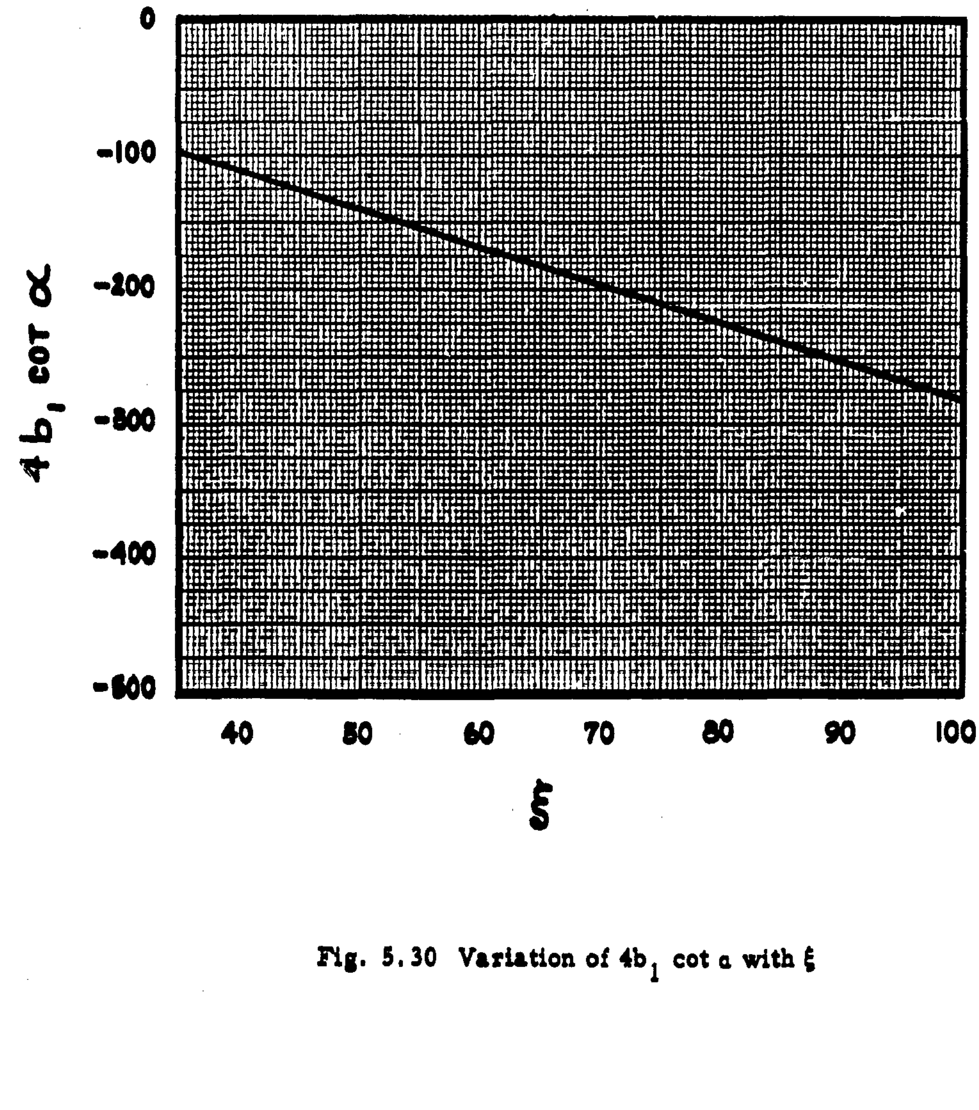


Fig. 5.30 Variation of $4b_1 \cot \alpha$ with ξ

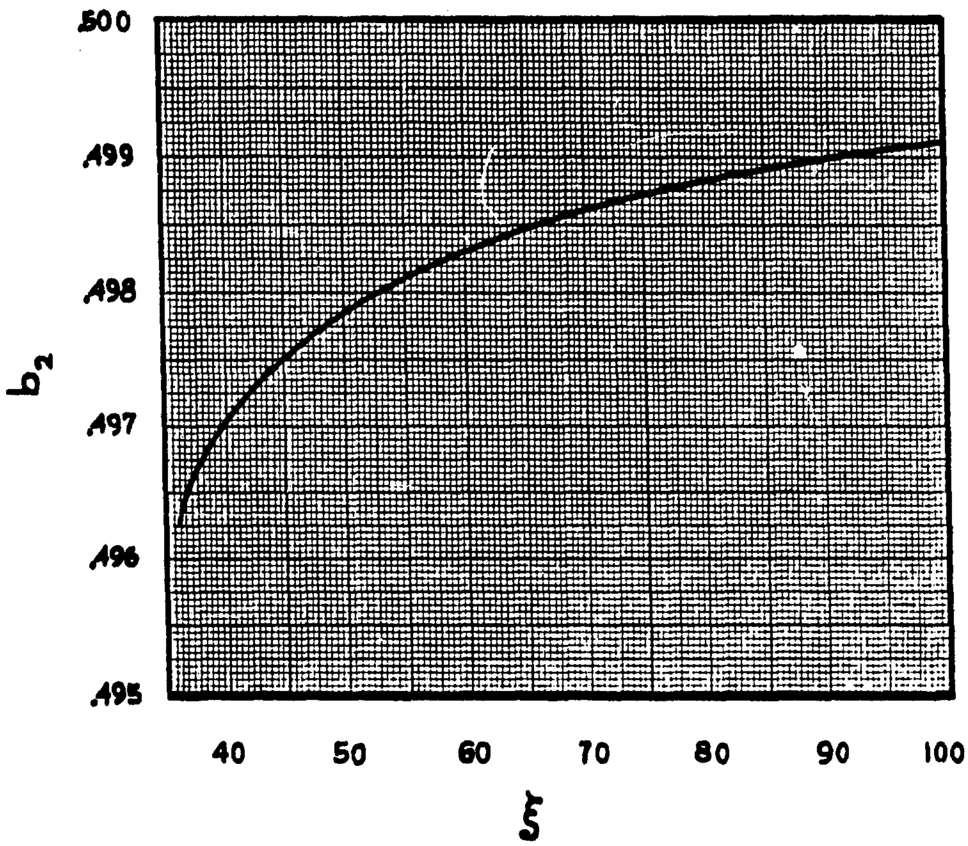


Fig. 5.31 Variation of b_2 with ξ

REFERENCES

1. Galletly, G. D., "Influence Coefficients and Pressure Vessel Analysis," Journal of Engineering for Industry, August 1960.

2. Watts, G. W., and Lang, H. A., "Stresses in a Pressure Vessel with a Conical Head," Trans. ASME, 1952.

3. Taylor, C. E., and Wenk, E., "Analysis of Stresses in the Conical Elements of Shell Structures," Proceedings of the Second U.S. National Congress of Applied Mechanics, ASME, New York, 1955.

4. Timoshenko, S., and Woinowsky-Krieger, S., "Theory of Plates and Shells" (McGraw-Hill Book Co., Inc., New York, 1959), Second Edition.

5. Watts, G. W., and Lang, H. A., "The Stresses in a Pressure Vessel with a Flat Head Closure," ASME Collected Papers (1927-1959) on Pressure Vessel and Piping Design.

6. Au, N. N., "Discontinuity Stresses in Pressure Vessels-Short Circular Cylindrical Shell with Ellipsoidal Head Closure, Part 2-Shells of Unequal Thickness," 31 August 1961; Aerospace Corporation Report No. TDR-594(1108)TN-2.

7. Lowell, H. H., "Tables of the Bessel-Kelvin Functions Ber, Bei, Ker, Kei, and their Derivatives for the Argument Range O (0.01) 107.50," NASA TR R-32, 1959.

8. Nosova, L. N., "Tables of Thomson Functions and their First Derivatives," (Pergamon Press, New York, 1961).

9. Flügge, W., "Stresses in Shells," Springer-Verlag, Berlin/Göttinger/Heidelberg, 1960.

10. Jones, R. H. and Orange, T. W., "Theoretical Elastic Stress Distributions Arising from Discontinuities and Edge Loads in Several Shell-Type Structures," NASA TR R-103, 1961.

APPENDIX

DERIVATION OF FORMULAS

A. BENDING STRESSES IN SHELLS

1. Symmetrical Deformation of Circular Cylindrical Shells

The differential equation governing the symmetrical deformation of circular cylindrical shells in accordance with the classical shell theory is

$$\frac{d^2}{dx^2}\left(D\frac{d^2 u_r}{dx^2}\right) + \frac{Eh}{a^2} u_r = Z \qquad \text{(A.1.1)}$$

where D = flexural rigidity of the shell

$$= \frac{Eh^3}{12(1-\nu^2)} \qquad \text{(A.1.2)}$$

Z = load intensity

h = thickness of shell

a = mean radius of curvature

E = Young's modulus of elasticity

ν = Poisson's ratio

u_r = radial displacement

In the case of uniform shell thickness, Eq. (A.1.1) becomes

$$D\frac{d^4 u_r}{dx^4} + \frac{Eh}{a^2} u_r = Z \qquad \text{(A.1.3)}$$

Using the notation

$$\beta^4 = \frac{Eh}{4a^2D} = \frac{3(1 - \nu^2)}{a^2h^2} \qquad \text{(A.1.4)}$$

Equation (A.1.3) can be represented in the simplified form

$$\frac{d^4 u_r}{dx^4} + 4\beta^4 u_r = \frac{Z}{D} \qquad \text{(A.1.5)}$$

for which the solution is

$$u_r = e^{\beta x}(C_1 \cos \beta x + C_2 \sin \beta x) + e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) + f(x) \qquad \text{(A.1.6)}$$

In Eq. (A.1.6), $f(x)$ is a particular solution and C_1, C_2, C_3, and C_4 are the constants of integration which must be determined in each particular case from the conditions at the ends of the cylinder.

The expressions for the membrane forces, bending moments, and transverse shearing force associated with the symmetrical deformation of circular cylindrical shells are

$$N_x = \frac{Eh}{1 - \nu^2}\left(\frac{du_x}{dx} - \nu\frac{u_r}{a}\right)$$

$$N_\theta = \frac{Eh}{1 - \nu^2}\left(-\frac{u_r}{a} + \nu\frac{du_x}{dx}\right)$$

$$M_x = -D\frac{d^2 u_r}{dx^2}$$

(cont.)

$$M_\theta = -\nu D \frac{d^2 u_r}{dx^2}$$

$$Q_x = -D \frac{d^3 u_r}{dx^3} \tag{A.1.7}$$

In the case where there is no pressure Z distributed over the surface of the shell and if the end condition is such that $N_x = 0$, then $f(x) = 0$ in Eq. (A.1.6). Furthermore, for long cylindrical shells subjected to loading conditions such that for large positive values of x, the deflection is finite, $C_1 = C_2 = 0$ and Eq. (A.1.6) reduces to

$$u_r = e^{-\beta x}(C_3 \cos \beta x + C_4 \sin \beta x) \tag{A.1.8}$$

For $N_x = 0$, the first expression of Eq. (A.1.7) gives

$$\frac{du_x}{dx} = \nu \frac{u_r}{a}$$

which when substituted into the expression for N_θ yields

$$N_\theta = -\frac{Eh u_r}{a} \tag{A.1.9}$$

The stresses in the shell are given by the expressions

$$\sigma_x = \pm \frac{6M_x}{h^2}$$

$$\sigma_\theta = \frac{N_\theta}{h} \pm \frac{6M_\theta}{h^2} \tag{A.1.10}$$

where σ_x = meridional stress

σ_θ = circumferential stress

2. Coordinate System and Signs Convention

The following signs convention for the rotation and deflection are used throughout this report:

Positive deflection - radially inward with respect to the center-line of the cylinder

Positive rotation - clockwise viewing the upper cut of the cylinder.

The signs convention for the forces and moments are:

Positive shearing force - upward on the left end and downward on the right end of the shell element, the left end being at the plane of the origin.

Positive moment - tensile on the inner surface or compressive on the outer surface with respect to the center-line of the cylinder.

The above signs are graphically represented in Fig. A.2.1.

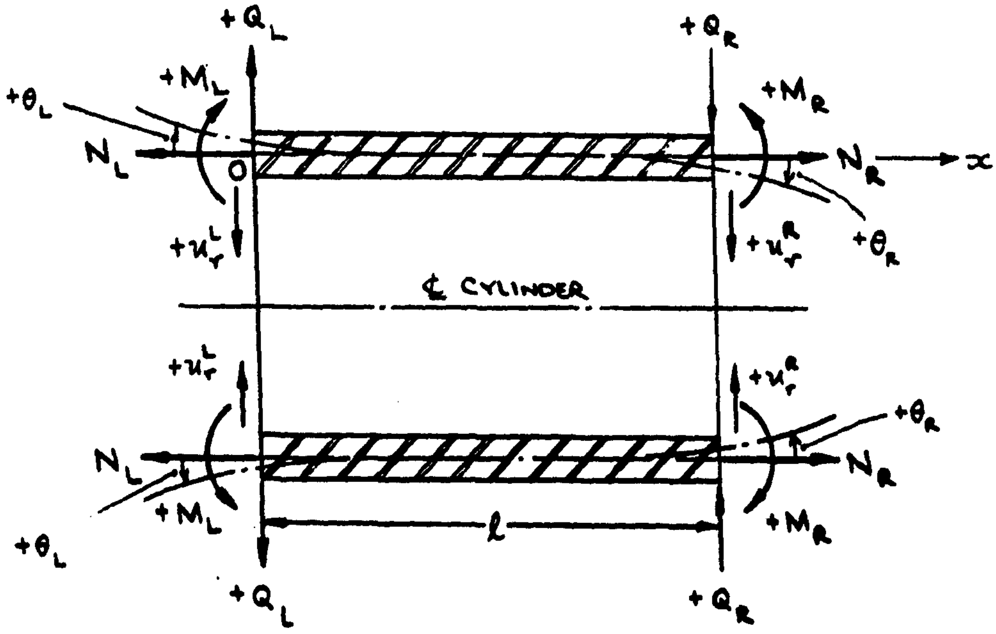
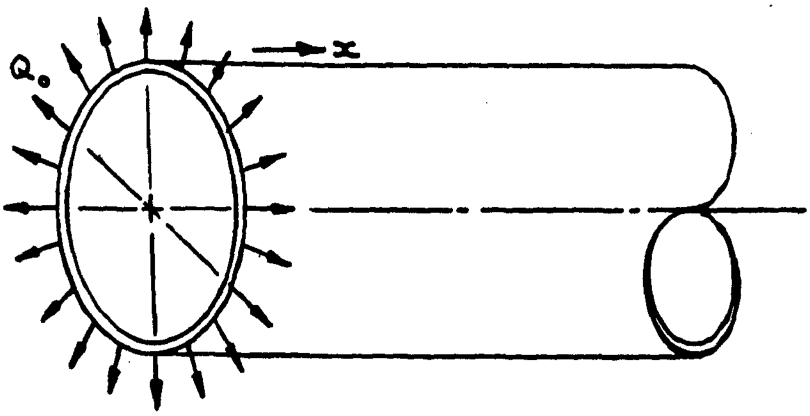


Fig. A.2.1 - Coordinate System and Signs Convention

3. <u>Long Circular Cylindrical Shell under the Action of Uniform Radial Shear Q_0 lb/in. of Circumference at End</u>



In the case of a long circular cylindrical shell under the action of uniform radial shear Q_o lb/in. along the circumference at the end, the two constants of integration C_3 and C_4 in Eq. (A.1.8) are determined from the conditions at the loaded end; viz.,

$$\left(M_x\right)_{x=0} = -D\left(\frac{d^2 u_r}{dx^2}\right)_{x=0} = 0 \tag{A.3.1}$$

$$\left(Q_x\right)_{x=0} = -D\left(\frac{d^3 u_r}{dx^3}\right)_{x=0} = Q_o \tag{A.3.2}$$

Specifically, they are

$$C_3 = -\frac{Q_o}{2\beta^3 D}$$

$$C_4 = 0 \tag{A.3.3}$$

These values of C_3 and C_4 may now be substituted into Eq. (A.1.8) to give

$$u_r = -\frac{Q_o e^{-\beta x} \cos \beta x}{2\beta^3 D} \tag{A.3.4}$$

The successive differentiations of Eq. (A.3.4) are

$$\frac{du_r}{dx} = \frac{Q_o e^{-\beta x}}{2\beta^2 D} (\sin \beta x + \cos \beta x)$$

(cont.)

$$\frac{d^2 u_r}{dx^2} = - \frac{Q_o e^{-\beta x} \sin \beta x}{\beta D}$$

$$\frac{d^3 u_r}{dx^3} = - \frac{Q_o e^{-\beta x}}{D} (\cos \beta x - \sin \beta x) \qquad \text{(A. 3. 5)}$$

The slope is therefore given by

$$\theta = \frac{du_r}{dx} = \frac{Q_o}{2\beta^2 D} e^{-\beta x} (\sin \beta x + \cos \beta x) \qquad \text{(A. 3. 6)}$$

and at the loaded end, $x = 0$; therefore,

$$\theta_{(x=0)} = \frac{Q_o}{2\beta^2 D} \qquad \text{(A. 3. 7)}$$

The expressions for the bending moments M_x, M_θ, and the transverse shear force Q_x are, in accordance with Eqs. (A. 1. 7), respectively,

$$M_x = \frac{Q_o}{\beta} e^{-\beta x} \sin \beta x$$

$$M_\theta = \frac{\nu Q_o}{\beta} e^{-\beta x} \sin \beta x$$

$$Q_x = Q_o e^{-\beta x} (\cos \beta x - \sin \beta x) \qquad \text{(A. 3. 8)}$$

and the hoop force, N_θ, according to Eqs. (A. 1. 9), (A. 3. 4), and (A. 1. 4) becomes

$$N_\theta = 2Q_o \beta a e^{-\beta x} \cos \beta x \qquad \text{(A. 3. 9)}$$

The expressions for the stresses in the shell are therefore

$$\sigma_x = \pm \frac{6Q_o}{\beta h^2} e^{-\beta x} \sin \beta x$$

$$\sigma_\theta = \frac{2Q_o}{h} e^{-\beta x} (\beta a \cos \beta x \pm \frac{3\nu}{\beta h} \sin \beta x) \qquad \text{(A.3.10)}$$

The maximum deflection is at the loaded end, where

$$u_{r_{max}} = u_{r(x=0)} = -\frac{Q_o}{2\beta^3 D} \qquad \text{(A.3.11)}$$

Let us now define

$$\Omega_1 = e^{-\beta x} \sin \beta x \qquad \text{(A.3.12)}$$

$$\Omega_2 = e^{-\beta x} \cos \beta x \qquad \text{(A.3.13)}$$

$$\Omega_3 = e^{-\beta x} (\sin \beta x + \cos \beta x) \qquad \text{(A.3.14)}$$

Then, Eqs. (A.3.10) become

$$\sigma_x = \pm \frac{6Q_o}{\beta h^2} \Omega_1$$

$$\sigma_\theta = \frac{2Q_o}{h} \left(\beta a \Omega_2 \pm \frac{3\nu}{\beta h} \Omega_1\right) \qquad \text{(A.3.15)}$$

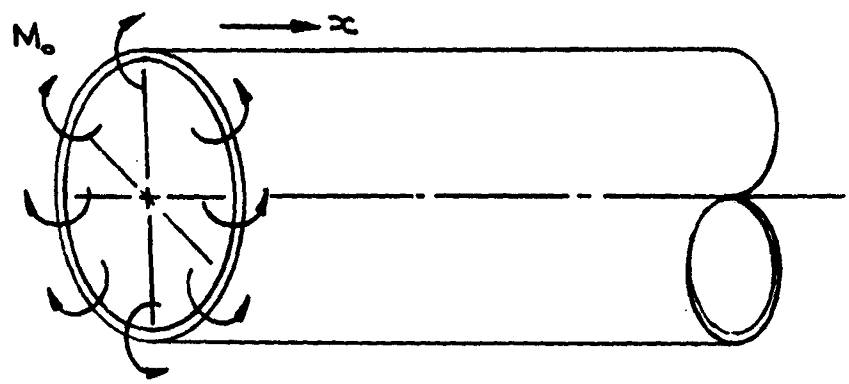
and Eqs. (A.3.4), (A.3.6) will become, respectively,

$$u_r = -\frac{Q_o}{2\beta^3 D}\Omega_2 \tag{A.3.16}$$

$$\theta = \frac{Q_o}{2\beta^2 D}\Omega_3 \tag{A.3.17}$$

In Eq. (A.3.15) where ± signs occur, the upper sign refers to the stress state at the inner surface of the cylinder and the lower sign to the stress state at the outer surface of the cylinder.

4. <u>Long Circular Cylindrical Shell under the Action of Uniform Radial Moment M_o in.-lb/in. of Circumference at End</u>



In this case, the constants C_3 and C_4 in Eq. (A.1.8) are determined from the following boundary conditions:

$$\left(M_x\right)_{x=0} = -D\left(\frac{d^2u_r}{dx^2}\right)_{x=0} = M_o$$

$$\left(Q_x\right)_{x=0} = -D\left(\frac{d^3u_r}{dx^3}\right)_{x=0} = 0 \tag{A.4.1}$$

Upon application of these boundary conditions to Eq. (A.1.8), there results

$$C_3 = -\frac{M_o}{2\beta^2 D}$$

$$C_4 = \frac{M_o}{2\beta^2 D} \tag{A.4.2}$$

With these values for C_3 and C_4, the expression for the deflection U_r becomes

$$u_r = \frac{M_o}{2\beta^2 D} e^{-\beta x} (\sin \beta x - \cos \beta x) \tag{A.4.3}$$

The successive differentiations of Eq. (A.4.3) are

$$\frac{du_r}{dx} = \frac{M_o}{\beta D} e^{-\beta x} \cos \beta x$$

$$\frac{d^2 u_r}{dx^2} = -\frac{M_o}{D} e^{-\beta x} (\sin \beta x + \cos \beta x)$$

$$\frac{d^3 u_r}{dx^3} = \frac{2M_o}{D} \beta e^{-\beta x} \sin \beta x \tag{A.4.4}$$

giving for the slope

$$\theta = \frac{du_r}{dx} = \frac{M_o}{\beta D} e^{-\beta x} \cos \beta x \tag{A.4.5}$$

The slope at the loaded end where x = 0 is, therefore,

$$\theta_{(x=0)} = \frac{M_o}{\beta D} \tag{A.4.6}$$

Equations (A.4.3) and (A.4.4) may now be introduced into Eqs. (A.1.7) and (A.1.9) to give the following expressions for M_x, M_θ, Q_x and N_θ. Indeed,

$$M_x = M_o e^{-\beta x} (\sin \beta x + \cos \beta x)$$

$$M_\theta = \nu M_o e^{-\beta x} (\sin \beta x + \cos \beta x)$$

$$Q_x = -2M_o \beta e^{-\beta x} \sin \beta x$$

$$N_\theta = 2M_o \beta^2 a e^{-\beta x} (\cos \beta x - \sin \beta x)$$

The expressions for the stresses in the shell are, accordingly,

$$\sigma_x = \pm \frac{6M_o}{h^2} e^{-\beta x} (\sin \beta x + \cos \beta x)$$

$$\sigma_\theta = \frac{2M_o}{h} e^{-\beta x} \left[\beta^2 a (\cos \beta x - \sin \beta x) \pm \frac{3\nu}{h} (\sin \beta x + \cos \beta x)\right]$$

or, by introducing the notations of Eqs. (A.3.13) and (A.3.14), viz.,

$$\Omega_2 = e^{-\beta x} \cos \beta x \tag{A.3.13}$$

$$\Omega_3 = e^{-\beta x} (\sin \beta x + \cos \beta x) \tag{A.3.14}$$

and defining

$$\Omega_4 = e^{-\beta x} (\cos \beta x - \sin \beta x) \tag{A.4.7}$$

we find

$$\sigma_x = \pm \frac{6M_o}{h^2} \Omega_3 \tag{A.4.8}$$

$$\sigma_\theta = \frac{2M_o}{h} (\beta^2 a \Omega_4 \pm \frac{3\nu}{h} \Omega_3) \tag{A.4.9}$$

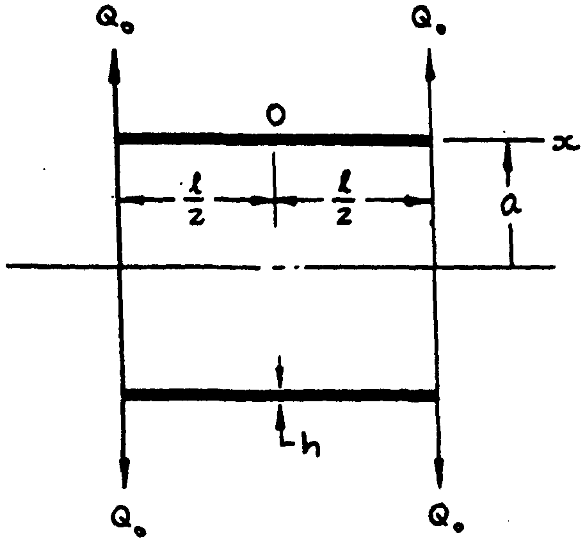
$$u_r = - \frac{M_o}{2\beta^2 D} \Omega_4 \tag{A.4.10}$$

and

$$\theta = \frac{M_o}{\beta D} \Omega_2 \tag{A.4.11}$$

5. Short Circular Cylindrical Shell Bent by Forces Distributed Along the Edges

(a) Equal Edge Forces.



In the case of shorter shells loaded such that no pressure Z is distributed over the surface of the shell, the deflection equation (A. 1. 6) can be put into the following form by the introduction of hyperbolic functions in place of the exponential functions. Thus,

$$U_r = C_1 \sin\beta x \sinh\beta x + C_2 \sin\beta x \cosh\beta x + C_3 \cos\beta x \sinh\beta x + C_4 \cos\beta x \cosh\beta x \quad \text{(A. 5. 1)}$$

By selecting the origin of coordinates to be at the middle of the cylinder, it is readily seen that Eq. (A. 5. 1) must be an even function of x. Accordingly,

$$C_2 = C_3 = 0 \quad \text{(A. 5. 2)}$$

and Eq. (A. 5. 1) reduces to

$$U_r = C_1 \sin\beta x \sinh\beta x + C_4 \cos\beta x \cosh\beta x \quad \text{(A. 5. 3)}$$

The constants C_1 and C_4 may now be determined from the conditions at the loaded end $x = \ell/2$ which may be written

$$\left(M_x\right)_{x=\ell/2} = -D\left(\frac{d^2 u_r}{dx^2}\right)_{x=\ell/2} = 0$$

$$\left(Q_x\right)_{x=\ell/2} = -D\left(\frac{d^3 u_r}{dx^3}\right)_{x=\ell/2} = -Q_o \tag{A.5.4}$$

Performing the above operations, there are obtained

$$C_1 = -\frac{Q_o}{\beta^3 D}\left(\frac{\sin\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2}}{\sin\beta\ell + \sinh\beta\ell}\right)$$

$$C_4 = -\frac{Q_o}{\beta^3 D}\left(\frac{\cos\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}}{\sin\beta\ell + \sinh\beta\ell}\right) \tag{A.5.5}$$

Equation (A.5.3) then becomes

$$u_r = -\frac{Q_o\left(\sin\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2}\sin\beta x\sinh\beta x + \cos\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}\cos\beta x\cosh\beta x\right)}{\beta^3 D\left(\sin\beta\ell + \sinh\beta\ell\right)} \tag{A.5.6}$$

The successive differentiations of Eq. (A.5.6) are

$$\frac{du_r}{dx} = -\frac{Q_o}{\beta^2 D}\left[\frac{\left(\sin\frac{\beta l}{2}\sinh\frac{\beta l}{2} + \cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\right)\cos\beta x \sinh\beta x + \left(\sin\frac{\beta l}{2}\sinh\frac{\beta l}{2} - \cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\right)\sin\beta x \cosh\beta x}{\sin\beta l + \sinh\beta l}\right]$$

$$\frac{d^2u_r}{dx^2} = -\frac{2Q_o}{\beta D}\left[\frac{\sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\cos\beta x\cosh\beta x - \cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\sin\beta x\sinh\beta x}{\sin\beta l + \sinh\beta l}\right]$$

$$\frac{d^3u_r}{dx^3} = -\frac{2Q_o}{D}\left[\frac{\sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\left(\cos\beta x\sinh\beta x - \sin\beta x\cosh\beta x\right) - \cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\left(\cos\beta x\sinh\beta x + \sin\beta x\cosh\beta x\right)}{\sin\beta l + \sinh\beta l}\right]$$

(A.5.7)

The deflection and the slope at the loaded ends are therefore, respectively,

$$u_{r(x=\pm\frac{l}{2})} = -\frac{Q_o}{2\beta^3 D}\left(\frac{\cosh\beta l + \cos\beta l}{\sin\beta l + \sinh\beta l}\right)$$

$$\theta_{(x=\pm\frac{l}{2})} = \left(\frac{du_r}{dx}\right)_{x=\pm\frac{l}{2}} = \mp\frac{Q_o}{2\beta^2 D}\left(\frac{\sinh\beta l - \sin\beta l}{\sinh\beta l + \sin\beta l}\right) \qquad \text{(A.5.8)}$$

and the expressions for M_x, M_θ, Q_x, and N_θ become, respectively,

$$M_x = -\frac{2Q_o}{\beta}\left[\frac{\cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\sin\beta x\sinh\beta x - \sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\cos\beta x\cosh\beta x}{\sin\beta l + \sinh\beta l}\right]$$

(cont.)

$$M_\theta = -\frac{2\nu Q_0}{\beta}\left[\frac{\cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\sin\beta x\sinh\beta x - \sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\cos\beta x\cosh\beta x}{\sin\beta l + \sinh\beta l}\right] \quad \text{(A.5.9)}$$

$$Q_x = -2Q_0\left[\frac{\cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\left(\cos\beta x\sinh\beta x + \sin\beta x\cosh\beta x\right) - \sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\left(\cos\beta x\sinh\beta x - \sin\beta x\cosh\beta x\right)}{\sin\beta l + \sinh\beta l}\right]$$

$$N_\theta = \frac{4Q_0\beta a\left(\sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\sin\beta x\sinh\beta x + \cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\cos\beta x\cosh\beta x\right)}{\sin\beta l + \sinh\beta l}$$

Hence the stresses in the shell are, respectively,

$$\sigma_x = \mp\frac{12Q_0}{\beta h^2}\left[\frac{\cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\sin\beta x\sinh\beta x - \sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\cos\beta x\cosh\beta x}{\sin\beta l + \sinh\beta l}\right] \quad \text{(A.5.10)}$$

$$\sigma_\theta = -\frac{4Q_0}{h\left(\sin\beta l + \sinh\beta l\right)}\left[\pm\frac{3\nu}{\beta h}\left(\cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\sin\beta x\sinh\beta x - \sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\cos\beta x\cosh\beta x\right)\right.$$
$$\left. - \beta a\left(\sin\frac{\beta l}{2}\sinh\frac{\beta l}{2}\sin\beta x\sinh\beta x + \cos\frac{\beta l}{2}\cosh\frac{\beta l}{2}\cos\beta x\cosh\beta x\right)\right]$$

Let us now define the following parameters:

$$\Omega_5 = \sin\beta x \sinh\beta x$$

$$\Omega_6 = \sin\beta x \cosh\beta x$$

$$\Omega_7 = \cos\beta x \sinh\beta x \qquad \text{(cont.)}$$

$$\Omega_8 = \cos \beta x \cosh \beta x$$

$$\Omega_9 = \frac{\sin \frac{\beta \ell}{2} \sinh \frac{\beta \ell}{2}}{\sin \beta \ell + \sinh \beta \ell} \qquad \text{(A. 5. 11a)}$$

$$\Omega_{12} = \frac{\cos \frac{\beta \ell}{2} \cosh \frac{\beta \ell}{2}}{\sin \beta \ell + \sinh \beta \ell}$$

$$\Omega_{13} = \frac{\sinh \beta \ell - \sin \beta \ell}{\sin \beta \ell + \sinh \beta \ell}$$

$$\Omega_{14} = \frac{\cosh \beta \ell + \cos \beta \ell}{\sin \beta \ell + \sinh \beta \ell} \qquad \text{(A. 5. 11b)}$$

Then the expressions for the stresses in the shell become, respectively,

$$\sigma_x = \pm \frac{12Q_o}{\beta h^2} (\Omega_8 \Omega_9 - \Omega_5 \Omega_{12}) \qquad \text{(A. 5. 12)}$$

$$\sigma_\theta = \frac{4Q_o}{h} \left[\beta a (\Omega_5 \Omega_9 + \Omega_8 \Omega_{12}) \pm \frac{3\nu}{\beta h} (\Omega_8 \Omega_9 - \Omega_5 \Omega_{12}) \right] \qquad \text{(A. 5. 13)}$$

and the expressions for deflection and slope are, respectively,

$$u_r = - \frac{Q_o}{\beta^3 D} (\Omega_5 \Omega_9 + \Omega_8 \Omega_{12}) \qquad \text{(A. 5. 14)}$$

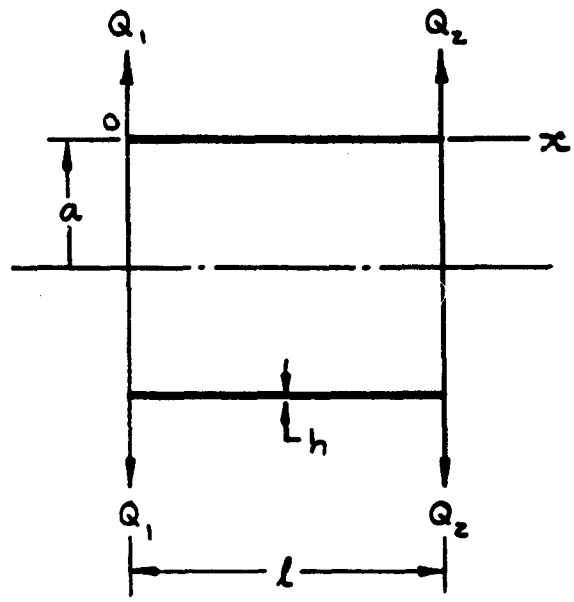
$$\theta = - \frac{Q_o}{\beta^3 D} \left[\Omega_7 (\Omega_9 + \Omega_{12}) + \Omega_6 (\Omega_9 - \Omega_{12}) \right] \qquad \text{(A. 5. 15)}$$

which become, at the loaded ends,

$$u_{r(x=\pm\frac{\ell}{2})} = - \frac{Q_o}{2\beta^3 D} \Omega_{14} \qquad \text{(A. 5. 16)}$$

$$\theta_{(x=\pm\frac{l}{2})} = \mp \frac{Q_o}{2\beta^2 D} \Omega_{13} \qquad \text{(A.5.17)}$$

(b) **Unequal Edge Forces**



In this case, the four constants of integration of Eq. (A. 5. 1) will be evaluated from the following boundary conditions:

$$\left(M_x\right)_{x=0} = -D\left(\frac{d^2u_r}{dx^2}\right)_{x=0} = 0$$

$$\left(M_x\right)_{x=l} = -D\left(\frac{d^2u_r}{dx^2}\right)_{x=l} = 0$$

$$\left(Q_x\right)_{x=0} = -D\left(\frac{d^3u_r}{dx^3}\right)_{x=0} = Q_1$$

$$\left(Q_x\right)_{x=l} = -D\left(\frac{d^3u_r}{dx^3}\right)_{x=l} = -Q_2 \qquad \text{(A.5.18)}$$

Specifically, they are

$$c_1 = 0$$

$$c_2 = \frac{\sin\beta l\,(Q_1 \sin\beta l - Q_2 \sinh\beta l)}{2\beta^3 D\,(\sinh^2\beta l - \sin^2\beta l)}$$

$$c_3 = \frac{\sinh\beta l\,(Q_1 \sinh\beta l - Q_2 \sin\beta l)}{2\beta^3 D\,(\sinh^2\beta l - \sin^2\beta l)}$$

$$c_4 = \frac{Q_1\,(\sin\beta l \cos\beta l - \sinh\beta l \cosh\beta l) + Q_2(\sin\beta l \cosh\beta l - \cos\beta l \sinh\beta l)}{2\beta^3 D\,(\sinh^2\beta l - \sin^2\beta l)}$$

(A.5.19)

These values of c_i ($i = 1, 2, 3, 4$) may now be substituted into Eq. (A.5.1) to give

$$u_r = -\frac{1}{2\beta^3 D(\sinh^2\beta l - \sin^2\beta l)}\Big\{(Q_2 \sinh\beta l - Q_1 \sin\beta l)\sin\beta l \sin\beta x \cosh\beta x + (Q_2 \sin\beta l - Q_1 \sinh\beta l)\sinh\beta l \cos\beta x \sinh\beta x + \big[Q_2(\cos\beta l \sinh\beta l - \sin\beta l \cosh\beta l) + Q_1(\sinh\beta l \cosh\beta l - \sin\beta l \cos\beta l)\big]\cos\beta x \cosh\beta x\Big\}$$

(A.5.20)

The successive differentiations of Eq. (A. 5. 20) are

$$\frac{du_r}{dx} = -\frac{1}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)}\Big\{\sin\beta l(Q_2\sinh\beta l - Q_1\sin\beta l)(\sin\beta x\sinh\beta x + \cos\beta x\cosh\beta x)$$
$$+ \sinh\beta l(Q_2\sin\beta l - Q_1\sinh\beta l)(\cos\beta x\cosh\beta x - \sin\beta x\sinh\beta x) +$$
$$(\cos\beta x\sinh\beta x - \sin\beta x\cosh\beta x)\Big[Q_2(\cos\beta l\sinh\beta l - \sin\beta l\cosh\beta l) +$$
$$Q_1(\sinh\beta l\cosh\beta l - \sin\beta l\cos\beta l)\Big]\Big\} \qquad \text{(A.5.21)}$$

$$\frac{d^2u_r}{dx^2} = -\frac{1}{\beta D(\sinh^2\beta l - \sin^2\beta l)}\Big\{\sin\beta l(Q_2\sinh\beta l - Q_1\sin\beta l)\cos\beta x\sinh\beta x -$$
$$\sinh\beta l(Q_2\sin\beta l - Q_1\sinh\beta l)\sin\beta x\cosh\beta x -$$
$$\Big[Q_2(\cos\beta l\sinh\beta l - \sin\beta l\cosh\beta l) + Q_1(\sinh\beta l\cosh\beta l - \sin\beta l\cos\beta l)\Big]\sin\beta x\sinh\beta x\Big\}$$

$$\frac{d^3u_r}{dx^3} = -\frac{1}{D(\sinh^2\beta l - \sin^2\beta l)}\Big\{\sin\beta l(Q_2\sinh\beta l - Q_1\sin\beta l)(\cos\beta x\cosh\beta x - \sin\beta x\sinh\beta x) -$$
$$\sinh\beta l(Q_2\sin\beta l - Q_1\sinh\beta l)(\sin\beta x\sinh\beta x + \cos\beta x\cosh\beta x) .$$
$$-(\cos\beta x\sinh\beta x + \sin\beta x\cosh\beta x)\Big[Q_2(\cos\beta l\sinh\beta l - \sin\beta l\cosh\beta l) +$$
$$Q_1(\sinh\beta l\cosh\beta l - \sin\beta l\cos\beta l)\Big]\Big\}$$

The deflection and the slope at the loaded end where x = 0 are then

$$u_{r(x=0)} = -\frac{Q_2(\cos\beta l\sinh\beta l - \sin\beta l\cosh\beta l) + Q_1(\sinh\beta l\cosh\beta l - \sin\beta l\cos\beta l)}{2\beta^3 D(\sinh^2\beta l - \sin^2\beta l)} \qquad \text{(A.5.22)}$$

$$\theta_{(x=0)} = -\frac{2Q_2 \sin\beta l \sinh\beta l - Q_1(\sin^2\beta l + \sinh^2\beta l)}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)} \qquad \text{(A.5.23)}$$

On the other hand, the deflection and the slope at the loaded end where $x = l$ will be

$$u_{r(x=l)} = -\frac{Q_2(\sinh\beta l \cosh\beta l - \sin\beta l \cos\beta l) + Q_1(\cos\beta l \sinh\beta l - \sin\beta l \cosh\beta l)}{2\beta^3 D(\sinh^2\beta l - \sin^2\beta l)} \qquad \text{(A.5.24)}$$

$$\theta_{(x=l)} = -\frac{Q_2(\sin^2\beta l + \sinh^2\beta l) - 2Q_1 \sin\beta l \sinh\beta l}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)} \qquad \text{(A.5.25)}$$

The expressions for M_x, M_θ, Q_x, and N_θ may now be easily obtained by substituting the expressions for u_r, d^2u_r/dx^2, and d^3u_r/dx^3 from Eqs. (A. 5. 20) and (A. 5. 21) into Eq. (A. 1. 7). Of particular interest to our study are the stresses in the shell. These are found to be

$$\sigma_x = \pm \frac{6}{\beta h^2(\sinh^2\beta l - \sin^2\beta l)} \Big\{ \sin\beta l(Q_2 \sin\beta l - Q_1 \sinh\beta l)\cos\beta x \sinh\beta x -$$

$$\sinh\beta l(Q_2 \sin\beta l - Q_1 \sinh\beta l)\sin\beta x \cosh\beta x -$$

$$\left[Q_2(\cos\beta l \sinh\beta l - \sin\beta l \cosh\beta l) + Q_1(\sinh\beta l \cosh\beta l - \sin\beta l \cos\beta l)\right]\sin\beta x \sinh\beta x$$

$$
\sigma_\theta = \frac{z}{h(\sinh^2\beta\ell - \sin^2\beta\ell)} \Bigg\{ \sin\beta\ell\left(Q_2\sinh\beta\ell - Q_1\sin\beta\ell\right)\left(\beta a\sin\beta x\cosh\beta x \pm \frac{3\nu}{\beta h}\cos\beta x\sinh\beta x\right)
$$

$$
+ \sinh\beta\ell\left(Q_2\sin\beta\ell - Q_1\sinh\beta\ell\right)\left(\beta a\cos\beta x\sinh\beta x \mp \frac{3\nu}{\beta h}\sin\beta x\cosh\beta x\right)
$$

$$
+ \left(\beta a\cos\beta x\cosh\beta x \mp \frac{3\nu}{\beta h}\sin\beta x\sinh\beta x\right)\Big[Q_2\left(\cos\beta\ell\sinh\beta\ell - \sin\beta\ell\cosh\beta\ell\right) +
$$

$$
Q_1\left(\sinh\beta\ell\cosh\beta\ell - \sin\beta\ell\cos\beta\ell\right)\Big]\Bigg\}
$$

Using the parameters Ω_5, Ω_6, Ω_7, Ω_8 defined in Eq. (A. 5. 11) and introducing the following additional parameters:

$$
\Omega_{16} = \frac{\sinh\beta\ell\cosh\beta\ell - \sin\beta\ell\cos\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}
$$

$$
\Omega_{18} = \frac{\cos\beta\ell\sinh\beta\ell - \sin\beta\ell\cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}
$$

$$
\Omega_{20} = \frac{\sinh^2\beta\ell + \sin^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}
$$

$$
\Omega_{21} = \frac{\sin\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}
$$

$$
\Omega_{22} = \frac{\sin^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}
$$

$$
\Omega_{23} = \frac{\sinh^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell} \qquad \text{(A.5.26)}
$$

We find the expressions for the stresses in the shell to be

$$\sigma_x = \pm \frac{6}{\beta h^2}\Big[(\Omega_{22}Q_2 - \Omega_{21}Q_1)\Omega_7 - (\Omega_{21}Q_2 - \Omega_{22}Q_1)\Omega_6 - (\Omega_{18}Q_2 + \Omega_{16}Q_1)\,\Omega_5\Big] \tag{A.5.27}$$

$$\sigma_\theta = \frac{2}{h}\Big\{\beta a\Big[\Omega_6(\Omega_{21}Q_2 - \Omega_{22}Q_1) + \Omega_7(\Omega_{21}Q_2 - \Omega_{23}Q_1) + \Omega_8(\Omega_{18}Q_2 + \Omega_{16}Q_1)\Big] \pm \frac{3\nu}{\beta h}\Big[\Omega_7(\Omega_{21}Q_2 - \Omega_{22}Q_1) - \Omega_6(\Omega_{21}Q_2 - \Omega_{23}Q_1) - \Omega_5(\Omega_{18}Q_2 + \Omega_{16}Q_1)\Big]\Big\} \tag{A.5.28}$$

The expressions for deflection and slope are correspondingly

$$u_r = -\frac{1}{2\beta^3 D}\Big[(\Omega_{21}Q_2 - \Omega_{22}Q_1)\,\Omega_6 + (\Omega_{21}Q_2 - \Omega_{23}Q_1)\,\Omega_7 + (\Omega_{18}Q_2 + \Omega_{16}Q_1)\,\Omega_8\Big] \tag{A.5.29}$$

$$\theta = -\frac{1}{2\beta^2 D}\Big[(\Omega_{21}Q_2 - \Omega_{22}Q_1)(\Omega_5 + \Omega_8) + (\Omega_{21}Q_2 - \Omega_{23}Q_1)(\Omega_8 - \Omega_5) + (\Omega_{18}Q_2 + \Omega_{16}Q_1)(\Omega_7 - \Omega_6)\Big] \tag{A.5.30}$$

and at the loaded ends, the deflections and slopes become

$$u_{r\,(x=0)} = -\frac{1}{2\beta^3 D}\left(\Omega_{18}Q_2 + \Omega_{16}Q_1\right) \tag{A.5.31}$$

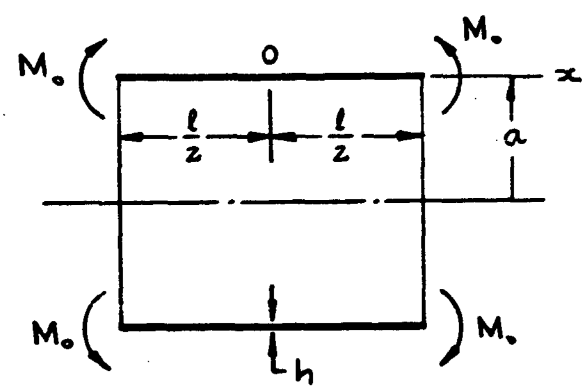
$$u_{r(x=l)} = -\frac{1}{2\beta^3 D}\left(\Omega_{16}Q_2 + \Omega_{18}Q_1\right) \tag{A.5.32}$$

$$\theta_{(x=0)} = -\frac{1}{2\beta^2 D}\left(2\,\Omega_{21}Q_2 - \Omega_{20}Q_1\right) \tag{A.5.33}$$

$$\theta_{(x=l)} = -\frac{1}{2\beta^2 D}\left(\Omega_{20}Q_2 - 2\Omega_{21}Q_1\right) \tag{A.5.34}$$

6. <u>Short Circular Cylindrical Shell Bent by Moments Distributed Along the Edges</u>

(a) Equal Edge Moments



Again, in this case, Eq. (A. 5. 3) is applicable. The constants c_1 and c_4 may be determined from the following boundary conditions:

$$M_x\left(x=\tfrac{l}{2}\right) = -D\left(\frac{d^2u_r}{dx^2}\right)_{x=\frac{l}{2}} = M_o$$

$$Q_x\left(x=\tfrac{l}{2}\right) = -D\left(\frac{d^3u_r}{dx^3}\right)_{x=\frac{l}{2}} = 0 \tag{A.6.1}$$

Specifically, they are

$$C_1 = -\frac{M_0}{\beta^2 D}\left(\frac{\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2} + \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}}{\sinh\beta l + \sin\beta l}\right)$$

$$C_4 = -\frac{M_0}{\beta^2 D}\left(\frac{\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2} - \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}}{\sinh\beta l + \sin\beta l}\right) \tag{A. 6. 2}$$

The deflection equation for the middle surface of the shell is then

$$u_r = -\frac{M_0\left[\left(\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2} + \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}\right)\sin\beta x\sinh\beta x + \left(\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2} - \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}\right)\cos\beta x\cosh\beta x\right]}{\beta^2 D\left(\sinh\beta l + \sin\beta l\right)} \tag{A. 6. 3}$$

The successive differentiations of Eq. (A. 6. 3) are

$$\frac{du_r}{dx} = -\frac{2M_0\left(\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2}\cos\beta x\sinh\beta x + \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}\sin\beta x\cosh\beta x\right)}{\beta D\left(\sinh\beta l + \sin\beta l\right)}$$

$$\frac{d^2u_r}{dx^2} = -\frac{2M_0\left[\left(\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2} + \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}\right)\cos\beta x\cosh\beta x - \left(\cos\frac{\beta l}{2}\sinh\frac{\beta l}{2} - \sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}\right)\sin\beta x\sinh\beta x\right]}{D\left(\sinh\beta l + \sin\beta l\right)}$$

$$\frac{d^3u_r}{dx^3} = -\frac{4\beta M_0\left(\sin\frac{\beta l}{2}\cosh\frac{\beta l}{2}\cos\beta x\sinh\beta x - \cos\frac{\beta l}{2}\sinh\frac{\beta l}{2}\sin\beta x\cosh\beta x\right)}{D\left(\sinh\beta l + \sin\beta l\right)} \tag{A. 6. 4}$$

The deflection and slope at the loaded ends are now obtained by substituting $\pm \ell/2$ for x in the above equations. Specifically, they are

$$u_r\left(x = \pm \frac{\ell}{2}\right) = -\frac{M_o}{2\beta^2 D}\left(\frac{\sinh\beta\ell - \sin\beta\ell}{\sinh\beta\ell + \sin\beta\ell}\right) \tag{A.6.5}$$

$$\theta_{\left(x = \pm \frac{\ell}{2}\right)} = \mp \frac{M_o}{\beta D}\left(\frac{\cosh\beta\ell - \cos\beta\ell}{\sinh\beta\ell + \sin\beta\ell}\right) \tag{A.6.6}$$

The stresses in the shell are, respectively,

$$\sigma_x = \pm \frac{12M_o\left[\left(\cos\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2} + \sin\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}\right)\cos\beta x\cosh\beta x - \left(\cos\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2} - \sin\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}\right)\sin\beta x\sinh\beta x\right]}{h^2(\sinh\beta\ell + \sin\beta\ell)}$$

$$\sigma_\theta = \frac{4M_o}{h(\sinh\beta\ell + \sin\beta\ell)}\left[\left(\cos\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2} + \sin\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}\right)\left(\beta^2 a\sin\beta x\sinh\beta x \pm \frac{3\nu}{h}\cos\beta x\cosh\beta x\right)\right.$$
$$\left. + \left(\cos\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2} - \sin\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}\right)\left(\beta^2 a\cos\beta x\cosh\beta x \mp \frac{3\nu}{h}\sin\beta x\sinh\beta x\right)\right]$$

Using the parameters Ω_5, Ω_6, Ω_7, Ω_8, Ω_{13} defined in Eq. (A.5.11), and introducing the following additional parameters:

$$\Omega_{10} = \frac{\sin\frac{\beta\ell}{2}\cosh\frac{\beta\ell}{2}}{\sin\beta\ell + \sinh\beta\ell}$$

$$\Omega_{11} = \frac{\cos\frac{\beta\ell}{2}\sinh\frac{\beta\ell}{2}}{\sin\beta\ell + \sinh\beta\ell}$$

(cont.)

$$\Omega_{15} = \frac{\cosh\beta l - \cos\beta l}{\sin\beta l + \sinh\beta l} \tag{A.6.7}$$

We find the stresses in the shell to be

$$\sigma_x = \pm \frac{12M_0}{h^2}\left[(\Omega_{11}+\Omega_{10})\Omega_8 - (\Omega_{11}-\Omega_{10})\Omega_5\right] \tag{A.6.8}$$

$$\sigma_\theta = \frac{4M_0}{h}\left\{\beta^2 a\left[\Omega_{11}(\Omega_5+\Omega_8)+\Omega_{10}(\Omega_5-\Omega_8)\right] \pm \frac{3\nu}{h}\left[\Omega_{10}(\Omega_5+\Omega_8)-\Omega_{11}(\Omega_5-\Omega_8)\right]\right\} \tag{A.6.9}$$

together with the following expressions for deflection and slope:

$$u_r = -\frac{M_0}{\beta^2 D}\left[(\Omega_{11}+\Omega_{10})\Omega_5 + (\Omega_{11}-\Omega_{10})\Omega_8\right] \tag{A.6.10}$$

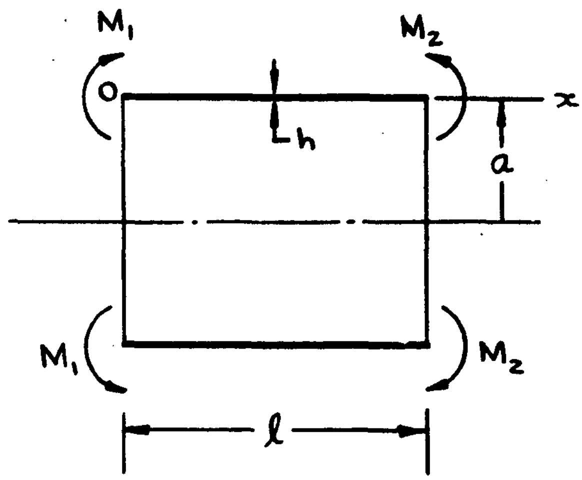
$$\theta = -\frac{2M_0}{\beta D}(\Omega_{11}\Omega_7 + \Omega_{10}\Omega_6) \tag{A.6.11}$$

The corresponding expressions for deflection and slope at the loaded ends are then, respectively,

$$u_r(x=\pm\tfrac{l}{2}) = -\frac{1}{2\beta^2 D} M_0 \Omega_B \tag{A.6.12}$$

$$\theta_{\left(x=\pm\frac{\ell}{2}\right)} = \mp \frac{1}{\beta D} M_o \Omega_{15} \tag{A.6.13}$$

(b) **Unequal Edge Moments**



Here, because of the unsymmetrical loading, Eq. (A.5.1) must be used, and the constants of integration will be determined from the following boundary conditions:

$$\left(M_x\right)_{x=0} = -D\left(\frac{d^2u_r}{dx^2}\right)_{x=0} = M_1$$

$$\left(M_x\right)_{x=\ell} = -D\left(\frac{d^2u_r}{dx^2}\right)_{x=\ell} = M_2$$

$$\left(Q_x\right)_{x=0} = -D\left(\frac{d^3u_r}{dx^3}\right)_{x=0} = 0$$

$$\left(Q_x\right)_{x=\ell} = -D\left(\frac{d^3u_r}{dx^3}\right)_{x=\ell} = 0 \tag{A.6.14}$$

Specifically, they are

$$C_1 = -\frac{M_1}{2\beta^2 D}$$

$$C_2 = C_3 = \frac{M_1(\sin\beta l\cos\beta l + \sinh\beta l\cosh\beta l) - M_2(\sin\beta l\cosh\beta l + \cos\beta l\sinh\beta l)}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)}$$

$$C_4 = \frac{2M_2\sin\beta l\sinh\beta l - M_1(\sinh^2\beta l + \sin^2\beta l)}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)} \qquad \text{(A.6.15)}$$

To recapitulate, the deflection Eq. (A.5.1) is

$$u_r = C_1\sin\beta x\sinh\beta x + C_2\sin\beta x\cosh\beta x + C_3\cos\beta x\sinh\beta x + C_4\cos\beta x\cosh\beta x$$

with successive derivatives of

$$\frac{du_r}{dx} = \beta\Big[C_1(\sin\beta x\cosh\beta x + \cos\beta x\sinh\beta x) + C_2(\sin\beta x\sinh\beta x + \cos\beta x\cosh\beta x) + C_3(\cos\beta x\cosh\beta x - \sin\beta x\sinh\beta x) + C_4(\cos\beta x\sinh\beta x - \sin\beta x\cosh\beta x)\Big]$$

$$\frac{d^2u_r}{dx^2} = 2\beta^2(C_1\cos\beta x\cosh\beta x + C_2\cos\beta x\sinh\beta x - C_3\sin\beta x\cosh\beta x - C_4\sin\beta x\sinh\beta x)$$

$$\frac{d^3u_r}{dx^3} = 2\beta^3\Big[C_1(\cos\beta x\sinh\beta x - \sin\beta x\cosh\beta x) + C_2(\cos\beta x\cosh\beta x - \sin\beta x\sinh\beta x) - C_3(\sin\beta x\sinh\beta x + \cos\beta x\cosh\beta x) - C_4(\sin\beta x\cosh\beta x + \cos\beta x\sinh\beta x)\Big]$$

The deflection and the slope at the loaded end where $x = 0$ are then

$$u_{r(x=0)} = C_4$$

$$= \frac{2M_2 \sin\beta\ell \sinh\beta\ell - M_1(\sinh^2\beta\ell + \sin^2\beta\ell)}{2\beta^2 D(\sinh^2\beta\ell - \sin^2\beta\ell)} \qquad \text{(A.6.16)}$$

$$\theta_{(x=0)} = \beta(C_2 + C_3)$$

$$= \frac{M_1(\sin\beta\ell\cos\beta\ell + \sinh\beta\ell\cosh\beta\ell) - M_2(\sin\beta\ell\cosh\beta\ell + \cos\beta\ell\sinh\beta\ell)}{\beta D(\sinh^2\beta\ell - \sin^2\beta\ell)}$$

(A.6.17)

and the deflection and slope at the loaded end where $x = \ell$ are

$$u_{r(x=\ell)} = C_1 \sin\beta\ell\sinh\beta\ell + C_2(\sin\beta\ell\cosh\beta\ell + \cos\beta\ell\sinh\beta\ell) + C_4\cos\beta\ell\cosh\beta\ell$$

$$= \frac{1}{2\beta^2 D}\left[\frac{2M_1\sin\beta\ell\sinh\beta\ell - M_2(\sinh^2\beta\ell + \sin^2\beta\ell)}{\sinh^2\beta\ell - \sin^2\beta\ell}\right] \qquad \text{(A.6.18)}$$

$$\theta_{(x=\ell)} = \beta\Big[C_1(\sin\beta\ell\cosh\beta\ell + \cos\beta\ell\sinh\beta\ell) + 2C_2\cos\beta\ell\cosh\beta\ell$$

$$+ C_4(\cos\beta\ell\sinh\beta\ell - \sin\beta\ell\cosh\beta\ell)\Big]$$

$$= \frac{1}{\beta D}\left[\frac{M_1(\sin\beta\ell\cosh\beta\ell + \cos\beta\ell\sinh\beta\ell) - M_2(\sin\beta\ell\cos\beta\ell + \sinh\beta\ell\cosh\beta\ell)}{\sinh^2\beta\ell - \sin^2\beta\ell}\right]$$

(A.6.19)

The stresses in the shell are, respectively,

$$\sigma_x = \mp \frac{12\beta^2 D}{h^2}\left(C_1 \cos\beta x \cosh\beta x + C_2 \cos\beta x \sinh\beta x - C_3 \sin\beta x \cosh\beta x - C_4 \sin\beta x \sinh\beta x\right)$$

$$= \pm \frac{6}{h^2}\left\{\left[\frac{M_1(\sinh\beta l \cosh\beta l + \sin\beta l \cos\beta l) - M_2(\cos\beta l \sinh\beta l + \sin\beta l \cosh\beta l)}{\sinh^2\beta l - \sin^2\beta l}\right] \times\right.$$

$$(\sin\beta x \cosh\beta x - \cos\beta x \sinh\beta x) + M_1 \cos\beta x \cosh\beta x -$$

$$\left.\left[\frac{M_1(\sinh^2\beta l + \sin^2\beta l) - 2M_2 \sin\beta l \sinh\beta l}{\sinh^2\beta l - \sin^2\beta l}\right] \sin\beta x \sinh\beta x\right\} \qquad \text{(A.6.20)}$$

$$\sigma_\theta = -\frac{E}{a}u_r \pm \nu\sigma_x$$

$$= \frac{2\beta^2 a}{h}\left\{M_1 \sin\beta x \sinh\beta x + \frac{M_1(\sinh^2\beta l + \sin^2\beta l) - 2M_2 \sin\beta l \sinh\beta l}{\sinh^2\beta l - \sin^2\beta l} \cos\beta x \cosh\beta x\right.$$

$$- \left[\frac{M_1(\sin\beta l \cos\beta l + \sinh\beta l \cosh\beta l) - M_2(\sin\beta l \cosh\beta l + \cos\beta l \sinh\beta l)}{\sinh^2\beta l - \sin^2\beta l}\right](\sin\beta x \cosh\beta x -$$

$$\left.\cos\beta x \sinh\beta x)\right\} \pm \nu\sigma_x \qquad \text{(A.6.21)}$$

Using the parameters Ω_5, Ω_6, Ω_7, Ω_8 defined in Eq. (A.5.11), and Ω_{20}, Ω_{21} defined in Eq. (A.5.26) and introducing the following additional parameters:

$$\Omega_{17} = \frac{\sinh\beta l \cosh\beta l + \sin\beta l \cos\beta l}{\sinh^2\beta l - \sin^2\beta l}$$

$$\Omega_{19} = \frac{\cos\beta l \sinh\beta l + \sin\beta l \cosh\beta l}{\sinh^2\beta l - \sin^2\beta l} \quad \text{(A.6.22)}$$

We find the stresses in the shell to be

$$\sigma_x = \pm \frac{6}{h^2}\Big[(M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 - \Omega_7) - (M_1\Omega_{20} - 2M_2\Omega_{21})\Omega_5 + M_1\Omega_8\Big] \quad \text{(A.6.23)}$$

$$\sigma_\theta = \frac{2}{h}\Big\{\beta^2 a\Big[M_1\Omega_5 - (M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 + \Omega_7) + \Omega_8(M_1\Omega_{20} - 2M_2\Omega_{21})\Big] \pm$$
$$\frac{3\nu}{h}\Big[(M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 - \Omega_7) - (M_1\Omega_{20} - 2M_2\Omega_{21})\Omega_5 + M_1\Omega_8\Big]\Big\} \quad \text{(A.6.24)}$$

The expression for the deflection and slope then becomes

$$u_r = \frac{1}{2\beta^2 D}\Big[(M_1\Omega_{17} - M_2\Omega_{19})(\Omega_6 + \Omega_7) - (M_1\Omega_{20} - 2M_2\Omega_{21})\Omega_8 - M_1\Omega_5\Big] \quad \text{(A.6.25)}$$

$$\theta = \frac{1}{2\beta D}\Big[2\Omega_8(M_1\Omega_{17} - M_2\Omega_{19}) - (M_1\Omega_{20} - 2M_2\Omega_{21})(\Omega_7 - \Omega_6)$$
$$- M_1(\Omega_6 + \Omega_7)\Big] \quad \text{(A.6.26)}$$

At the loaded ends, the expression for the deflection and the slope becomes

$$u_{r(x=0)} = -\frac{1}{2\beta^2 D}\left(M_1\Omega_{20} - 2M_2\Omega_{21}\right) \qquad \text{(A.6.27)}$$

$$u_{r(x=\ell)} = \frac{1}{2\beta^2 D}\left(2M_1\Omega_{21} - M_2\Omega_{20}\right) \qquad \text{(A.6.28)}$$

$$\theta_{(x=0)} = \frac{1}{\beta D}\left(M_1\Omega_{17} - M_2\Omega_{19}\right) \qquad \text{(A.6.29)}$$

$$\theta_{(x=\ell)} = \frac{1}{\beta D}\left(M_1\Omega_{19} - M_2\Omega_{17}\right) \qquad \text{(A.6.30)}$$

B. JUNCTURE SHEAR FORCES AND BENDING MOMENTS

1. Two Long Cylindrical Shells of Unequal Thicknesses

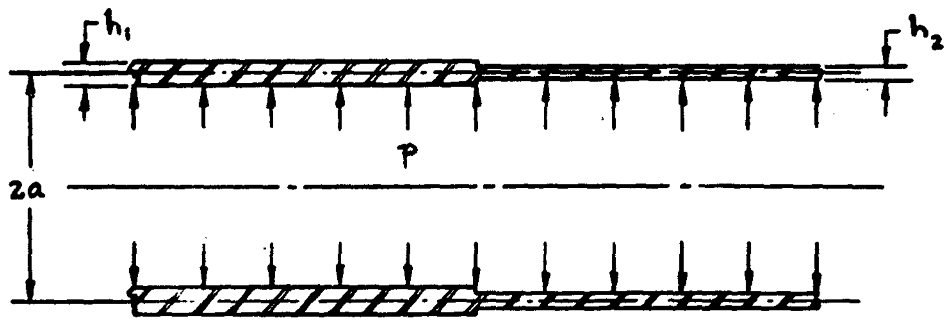


Fig. B.1.1. Two Long Circular Cylindrical Shells of Unequal Thickness Under Internal Pressure

Under the action of internal pressure, the circular cylindrical shells will extend radially, each by a different amount due to the different thicknesses. However, in the actual vessel, the two cylinders are kept together and strain compatibility at the junction indicates that there must act shearing forces Q_0 and bending moments M_0 uniformly distributed along the circumference and of such magnitudes as to eliminate this continuity.

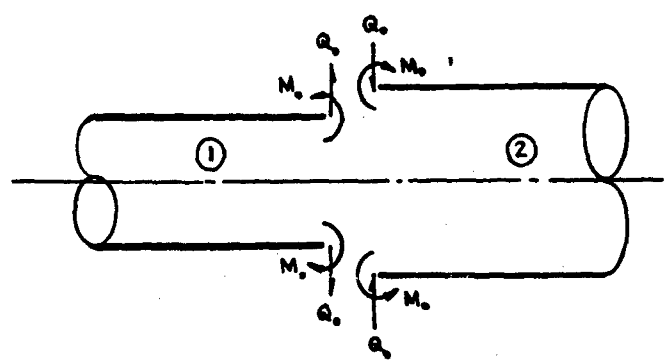


Fig. B.1.2. Forces and Moments at the Junction of the two Long Unequal Thickness Circular Cylindrical Shells

The stresses produced by these forces and moments are called <u>discontinuity stresses</u>. This condition is illustrated in Figure B.1.2.

To determine the magnitudes of Q_0 and M_0, it is necessary to have the appropriate formulas for long circular cylindrical shells bent by forces and moments distributed along the shell edges. These formulas are readily available in many texts on shells. Thus, from Reference 4, the deflection and rotation produced by the loads p, Q_0 and M_0 are:

(a) <u>For Cylinder No. 1:</u>

$$\delta = \frac{1}{2\beta_1^3 D_1}(\beta_1 M_0 + Q_0) + \frac{pa^2}{2Eh_1}(2-\nu) \qquad \text{(B.1.1)}$$

$$\delta' = \frac{1}{2\beta_1^2 D_1}(2\beta_1 M_0 + Q_0) \qquad \text{(B.1.2)}$$

(b) <u>For Cylinder No. 2:</u>

$$\delta = \frac{1}{2\beta_2^3 D_2}(\beta_2 M_0 - Q_0) + \frac{pa^2}{2Eh_2}(2-\nu) \qquad \text{(B.1.3)}$$

$$\delta' = \frac{1}{2\beta_2^2 D_2}(Q_0 - 2\beta_2 M_0) \qquad \text{(B.1.4)}$$

where δ = radial deflection of middle surface at junction, positive outward, inch.

δ' = rotation of middle surface at junction, positive as shown in Fig. B.1.3.

$$\beta_i^4 = \frac{Eh_i}{4a^2 D_i} = \frac{3(1 - \nu^2)}{a^2 h_i^2} \quad ; \quad (i = 1,2)$$

$$D_i = \frac{Eh_i^3}{12(1 - \nu^2)} \quad ; \quad (i = 1,2) \qquad \text{(B.1.5)}$$

a = mean radius of curvature of the circular cylinders;

p = internal pressure, psi;

E = modulus of elasticity, psi;

ν = Poisson's ratio

Q_o = uniformly distributed circumferential shearing force at the junction, lb per inch;

M_o = uniformly distributed circumferential bending moment at the junction, lb per inch;

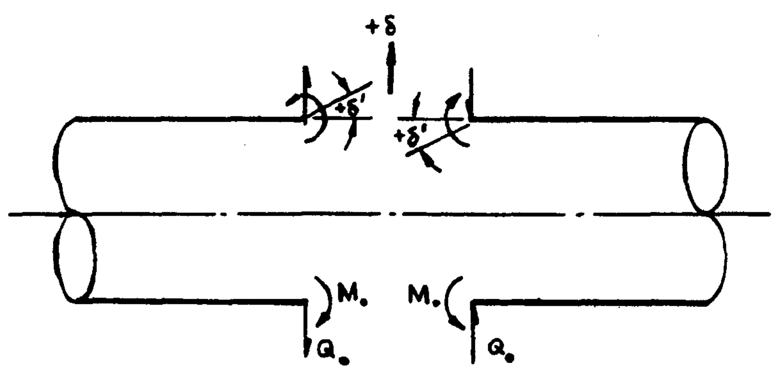


Fig. B.1.3 - Signs Convention for Middle Surface Radial Deflection and Rotation at Junction

For strain compatibility, we must have

δ (for Cylinder No. 1) = δ (for Cylinder No. 2)

δ' (for Cylinder No. 1) = δ' (for Cylinder No. 2) (B.1.6)

Hence

$$\frac{1}{2\beta_1^3 D_1}(\beta_1 M_0 + Q_0) + \frac{pa^2}{2Eh_1}(2-\nu) = \frac{1}{2\beta_2^3 D_2}(\beta_2 M_0 - Q_0) + \frac{pa^2}{2Eh_2}(2-\nu)$$

$$\frac{1}{2\beta_1^2 D_1}(2\beta_1 M_0 + Q_0) = \frac{1}{2\beta_2^2 D_2}(Q_0 - 2\beta_2 M_0)$$

from which

$$Q_0 = \frac{\frac{2pa^2}{E}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)(2-\nu)\left(\frac{1}{\beta_1 D_1}+\frac{1}{\beta_2 D_2}\right)}{\left(\frac{1}{\beta_1^2 D_1}+\frac{1}{\beta_2^2 D_2}\right)^2 + \frac{2}{\beta_1\beta_2 D_1 D_2}\left(\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}\right)} \tag{B.1.7}$$

$$M_0 = \frac{\frac{pa^2}{E}\left(\frac{1}{h_2}-\frac{1}{h_1}\right)(2-\nu)\left(\frac{1}{\beta_2^2 D_2}-\frac{1}{\beta_1^2 D_1}\right)}{\left(\frac{1}{\beta_1^2 D_1}+\frac{1}{\beta_2^2 D_2}\right)^2 + \frac{2}{\beta_1\beta_2 D_1 D_2}\left(\frac{1}{\beta_1^2}+\frac{1}{\beta_2^2}\right)} \tag{B.1.8}$$

These formulas for Q_o and M_o may be simplified to the following dimensionless form:

$$\frac{2\beta_1 Q_0}{p(2-\nu)} = \frac{(c-1)(c^{5/2}+1)}{(c^2+1)^2 + 2c^{3/2}(c+1)} \tag{B.1.9}$$

$$\frac{4\beta_1^2 M_0}{p(2-\nu)} = \frac{(c-1)(c^2-1)}{(c^2+1)^2 + 2c^{3/2}(c+1)} \tag{B.1.10}$$

where c = thickness ratio h_1/h_2.

2. Two Long Cylinders of Unequal Thicknesses and Mismatch of the Middle Surfaces

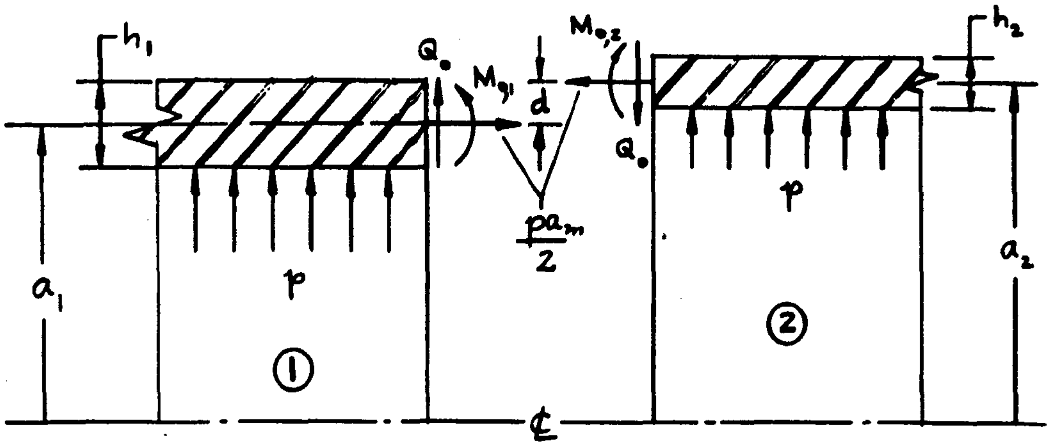


Fig. B.2.1 - Forces and Moments at the Junction of the Two Long Unequal Thickness Circular Cylindrical Shell with Mismatch of the Middle Surface

Analogous to the analysis of Case 1, we have

(a) For Cylinder No. 1:

$$\delta = \frac{1}{2\beta_1^3 D_1}\left(\beta_1 M_{o,1} + Q_o\right) + \frac{pa_1^2}{2Eh_1}(2-\nu) \qquad \text{(B.2.1)}$$

$$\delta' = \frac{1}{2\beta_1^2 D_1}\left(2\beta_1 M_{o,1} + Q_o\right) \qquad \text{(B.2.2)}$$

(b) For Cylinder No. 2:

$$\delta = \frac{1}{2\beta_2^3 D_2}\left(\beta_2 M_{o,2} - Q_o\right) + \frac{pa_2^2}{2Eh_2}(2-\nu) \qquad \text{(B.2.3)}$$

$$\delta' = \frac{1}{2\beta_2^2 D_2}(Q_0 - 2\beta_2 M_{0,2}) \tag{B.2.4}$$

where $M_{o,1}$ and $M_{o,2}$ are the uniformly distributed circumferential bending moments at the junction for cylinders 1 and 2, respectively, and are related to each other by the following equation:

$$M_{o,1} = M_{o,2} - \frac{1}{2}dpa_m \tag{B.2.5}$$

where

$$a_m = \frac{1}{2}(a_1 + a_2) \tag{B.2.6}$$

If we now substitute Eq. (B.2.5) into Eqs. (B.2.3) and (B.2.4), we obtain for cylinder No. 2,

$$\delta = \frac{1}{2\beta_2^3 D_2}\left[\beta_2\left(M_{0,1} + \frac{1}{2}dpa_m\right) - Q_0\right] + \frac{pa_2^2}{2Eh_2}(2-\nu) \tag{B.2.7}$$

$$\delta' = \frac{1}{2\beta_2^2 D_2}\left[Q_0 - 2\beta_2\left(M_{0,1} + \frac{1}{2}dpa_m\right)\right] \tag{B.2.8}$$

Again for strain compatibility, we must have

δ (for Cylinder No. 1) = δ (for Cylinder No. 2)
δ' (for Cylinder No. 1) = δ' (for Cylinder No. 2)

Hence,

$$\frac{1}{2\beta_1^3 D_1}(\beta_1 M_{o,1} + Q_o) + \frac{pa_1^2}{2Eh_1}(2-\nu) = \frac{1}{2\beta_2^3 D_2}\left[\beta_2\left(M_{o,1} + \frac{1}{2}dpa_m\right) - Q_o\right] + \frac{pa_2^2}{2Eh_2}(2-\nu)$$

$$\frac{1}{2\beta_1^2 D_1}(2\beta_1 M_{o,1} + Q_o) = \frac{1}{2\beta_2^2 D_2}\left[Q_o - 2\beta_2\left(M_{o,1} + \frac{1}{2}dpa_m\right)\right]$$

from which

$$Q_o = \frac{\dfrac{2-\nu}{E}\left(\dfrac{a_2^2}{h_2} - \dfrac{a_1^2}{h_1}\right)\left(\dfrac{1}{\beta_1 D_1} + \dfrac{1}{\beta_2 D_2}\right) + \dfrac{da_m}{2\beta_1\beta_2 D_1 D_2}\left(\dfrac{1}{\beta_1} + \dfrac{1}{\beta_2}\right)}{\dfrac{1}{2}\left(\dfrac{1}{\beta_1^2 D_1} + \dfrac{1}{\beta_2^2 D_2}\right)^2 + \dfrac{1}{\beta_1\beta_2 D_1 D_2}\left(\dfrac{1}{\beta_1^2} + \dfrac{1}{\beta_2^2}\right)}\, p \quad \text{(B.2.9)}$$

$$M_{o,1} = \frac{\dfrac{2-\nu}{2E}\left(\dfrac{a_2^2}{h_2} - \dfrac{a_1^2}{h_1}\right)\left(\dfrac{1}{\beta_2^2 D_2} - \dfrac{1}{\beta_1^2 D_1}\right) - \dfrac{da_m}{2\beta_2 D_2}\left(\dfrac{1}{\beta_1^3 D_1} + \dfrac{1}{2\beta_1^2\beta_2 D_1} + \dfrac{1}{2\beta_2^3 D_2}\right)}{\dfrac{1}{2}\left(\dfrac{1}{\beta_1^2 D_1} + \dfrac{1}{\beta_2^2 D_2}\right)^2 + \dfrac{1}{\beta_1\beta_2 D_1 D_2}\left(\dfrac{1}{\beta_1^2} + \dfrac{1}{\beta_2^2}\right)}\, p \quad \text{(B.2.10)}$$

Equations (B. 1a. 9) and (B. 1a. 10) may be put into the following convenient forms:

$$Q_o = \frac{\dfrac{2-\nu}{2\beta_1}(c-\ell^2)(c^{5/2}\ell^{-\frac{1}{2}} + 1) + da_m\beta_1 c^2\ell(c^{1/2}\ell^{1/2} + 1)}{(c^2+\ell)^2 + 2c^{3/2}\ell^{1/2}(c\ell+1)}\, p$$

$$M_{o,1} = \frac{\frac{2-\nu}{4\beta_1^2}(c-\ell^2)(c^2\ell^{-1}-1) - \frac{da_m c^2}{2}(c^2+2c^{1/2}\ell^{3/2}+\ell)}{(c^2+\ell)^2+2c^{3/2}\ell^{1/2}(c\ell+1)}\,p$$

$$M_{o,2} = M_{o,1} + \frac{1}{2}dpa_m$$

where c = thickness ratio h_1/h_2

ℓ = middle surface radius ratio a_1/a_2

Observe that the above equations are developed for $a_1 < a_2$. If $a_1 > a_2$, then d becomes negative, and the components of shear and bending moment due to the mismatch alone of the middle surfaces change in direction.

For thin shells, a/h is large and ℓ can be taken as unity for all practical purposes. The equations for Q_o, $M_{o,1}$ then simplify to

$$Q_o = \frac{\frac{2-\nu}{2\beta_1}(c-1)(c^{5/2}+1) + da_m\beta_1 c^2(c^{1/2}+1)}{(c^2+1)^2+2c^{3/2}(c+1)}\,p$$

$$M_{o,1} = \frac{\frac{2-\nu}{4\beta_1^2}(c-1)(c^2-1) + \frac{da_m c^2}{2}(c^2+2c^{1/2}+1)}{(c^2+1)^2+2c^{3/2}(c+1)}\,p$$

3. Long Circular Cylinder with Circumferential Ring Stiffeners

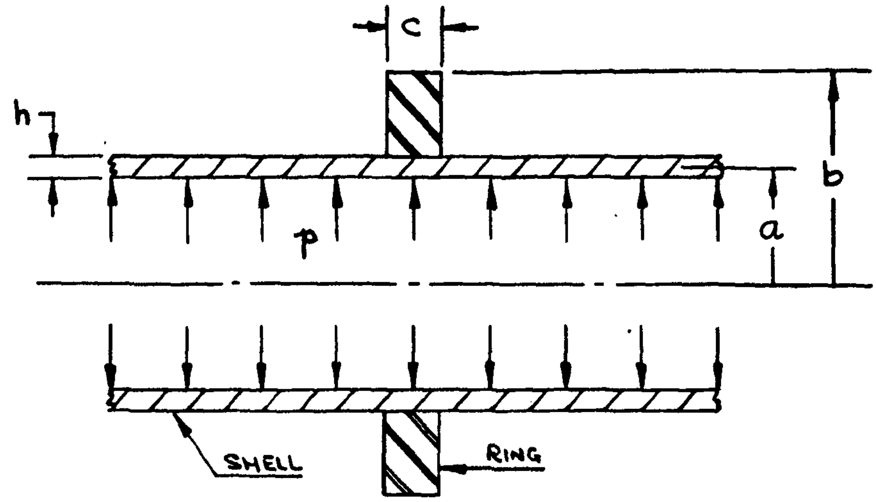


Fig. B.3.1 - Long Circular Cylinder with Circumferential Ring Stiffeners under Internal Pressure

Basic Assumptions: (a) The ring stiffener spacing is such that the influence of one does not extend to the next.

(b) The term "a" will be taken to be the outside of the shell instead of mean radius. For shells with a large radius to thickness ratio, the effect is negligible.

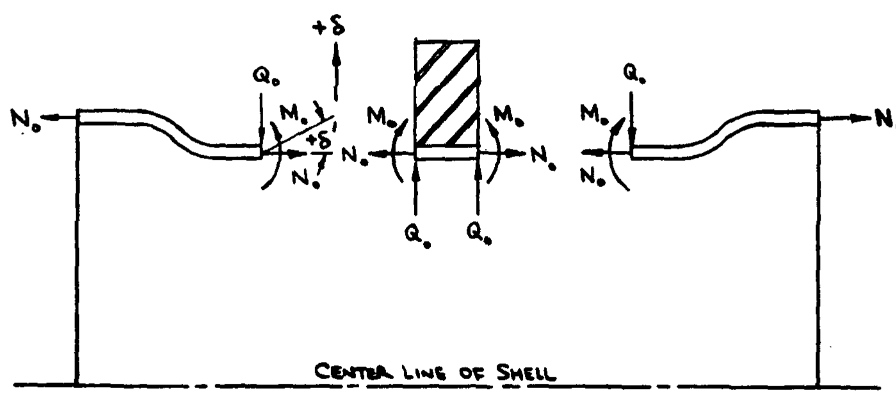


Fig. B.3.2 - Forces and Moments at the Junction of the Circular Cylindrical Shell and the Circumferential Ring Stiffener

Boundary Conditions: (a) The radial deflections of the shell and ring are equal at the junction.

(b) The slope of the shell adjacent to ring is zero.

Analysis: The deflection of the long circular cylindrical shell at the junction produced by the loads p, Q_o and M_o is, from Reference 4,

$$\delta = \frac{1}{2\beta^3 D}(\beta M_o - Q_o) + \frac{pa^2}{2Eh}(2 - \nu) \tag{B.3.1}$$

The radial displacement of the circumferential ring stiffener under the action of the forces $2Q_o$ and p may be readily obtained from the plane stress solution of a thick-walled cylinder under the action of internal pressure. Specifically, at the ring-shell interface,

$$\delta = \frac{a(2Q_o + pc)}{cE}\left(\frac{b^2 + a^2}{b^2 - a^2} + \nu\right) \tag{B.3.2}$$

In accordance with boundary condition (a), we can immediately write

$$\frac{1}{2\beta^3 D}(\beta M_o - Q_o) + \frac{pa^2}{2Eh}(2 - \nu) = \frac{a(2Q_o + pc)}{cE}\left(\frac{b^2 + a^2}{b^2 - a^2} + \nu\right) \tag{B.3.3}$$

The slope of the circular cylindrical shell adjacent to the ring produced by the forces p, Q_o and M_o is, from Reference 4,

$$\delta' = \frac{1}{2\beta^2 D}(2\beta M_o - Q_o) \tag{B.3.4}$$

This, in accordance with boundary condition (b) is zero. Hence

$$Q_o = 2\beta M_o \tag{B.3.5}$$

Equation (B.3.5) may now be substituted directly into Eq. (B.3.3) to give an explicit expression for the circumferential bending moment M_o. Specifically,

$$M_o = \frac{p\left[\frac{a(2-\nu)}{2h} - \left(\frac{b^2+a^2}{b^2-a^2}+\nu\right)\right]}{2\beta\left[\frac{a\beta}{h} + \frac{2}{c}\left(\frac{b^2+a^2}{b^2-a^2}+\nu\right)\right]} \qquad \text{(B.3.6)}$$

$$Q_o = 2\beta M_o \qquad \text{(B.3.7)}$$

4. Long Circular Cylindrical Shell with Many Equidistant Circumferential Ring Stiffeners

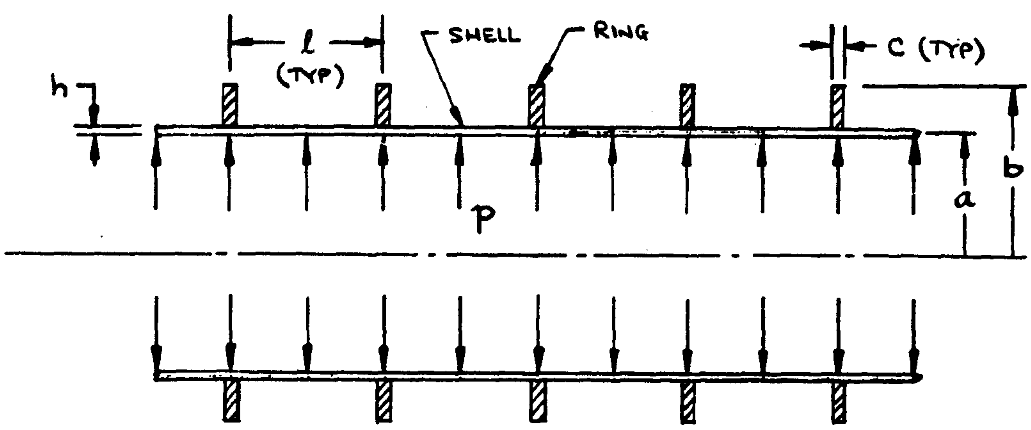


Fig. B.4.1 - Long Circular Cylindrical Shell with Many Equidistant Ring Stiffeners under Internal Pressure

Basic Assumptions: (a) The ring stiffener spacing is such that mutual influence becomes important.

(b) The rings are all of the same size and design.

(c) The term "a" will be taken to be the outside of the shell instead of mean radius. For shells with a large radius to thickness ratio, the effect is negligible.

Boundary Conditions: (a) The radial deflections of the shell and ring are equal at the junction.

(b) The slope of the shell adjacent to the ring is zero.

Analysis: As a consequence of the proximity of the ring stiffeners, the long cylinder must be treated as composed of a number of short cylinders, each of which has a length l. The discontinuity forces and moments at the junction of the short circular cylindrical shell and the circumferential ring stiffener may be taken to be identical to those shown in Fig. B.3.2. In this case, however, the deflection of the short circular cylindrical shell at the junction produced by the forces p, Q_o and M_o is, from Reference 4,

$$\delta = -\frac{2Q_o\beta a^2}{Eh}\left(\frac{\cosh\beta l + \cos\beta l}{\sinh\beta l + \sin\beta l}\right) + \frac{2M_o\beta^2 a^2}{Eh}\left(\frac{\sinh\beta l - \sin\beta l}{\sinh\beta l + \sin\beta l}\right) + \frac{pa^2}{2Eh}(2-\nu)$$

The radial displacement of the circumferential ring stiffener has been calculated previously and is

$$\delta = \frac{(2Q_o + pc)\,a}{cE}\left(\frac{b^2+a^2}{b^2-a^2} + \nu\right)$$

In accordance with boundary condition (a), we must have

$$-\frac{2Q_o\beta a^2}{Eh}\left(\frac{\cosh\beta l+\cos\beta l}{\sinh\beta l+\sin\beta l}\right)+\frac{2M_o\beta^2a^2}{Eh}\left(\frac{\sinh\beta l-\sin\beta l}{\sinh\beta l+\sin\beta l}\right)+\frac{pa^2}{2Eh}(2-\nu)$$
$$=\frac{(2Q_o+pc)a}{cE}\left(\frac{b^2+a^2}{b^2-a^2}+\nu\right) \tag{B.4.1}$$

The slope of the short circular cylindrical shell adjacent to the ring produced by the forces p, Q_o and M_o may also be obtained from Reference 4. Specifically, it is

$$\delta'=-\frac{2Q_o\beta^2a^2}{Eh}\left(\frac{\sinh\beta l-\sin\beta l}{\sinh\beta l+\sin\beta l}\right)+\frac{4M_o\beta^3a^2}{Eh}\left(\frac{\cosh\beta l-\cos\beta l}{\sinh\beta l+\sin\beta l}\right)$$

This, according to boundary condition (b) should be zero. Hence

$$Q_o=2\beta\left(\frac{\cosh\beta l-\cos\beta l}{\sinh\beta l-\sin\beta l}\right)M_o \tag{B.4.2}$$

If we now substitute Eq. (B.4.2) into Eq. (B.4.1), an explicit solution for the circumferential bending moment M_o will be obtained. Indeed,

$$M_o=\frac{\left[a\left(1-\frac{\nu}{2}\right)-h\left(\frac{b^2+a^2}{b^2-a^2}+\nu\right)\right]p}{4\beta\Omega_{15}\left[\frac{h}{c}\left(\frac{b^2+a^2}{b^2-a^2}+\nu\right)+\beta a\Omega_{14}\right]-2\beta^2a\Omega_{13}} \tag{B.4.3}$$

where
$$\Omega_{13} = \frac{\sinh \beta l - \sin \beta l}{\sinh \beta l + \sin \beta l}$$

$$\Omega_{14} = \frac{\cosh \beta l + \cos \beta l}{\sinh \beta l + \sin \beta l} \tag{B.4.4}$$

$$\Omega_{15} = \frac{\cosh \beta l - \cos \beta l}{\sinh \beta l - \sin \beta l}$$

Using the above notation, we find $Q_o = 2\beta\Omega_{15}M_o$ (B.4.5)

5. <u>Long Circular Cylindrical Shell with a Flat Head Closure</u>

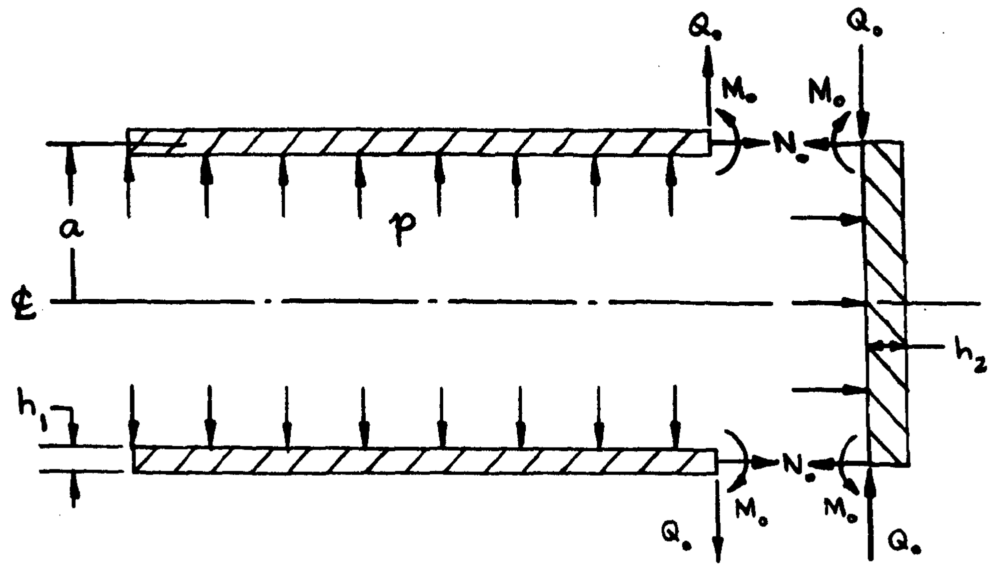


Fig. B.5.1 - Long Circular Cylindrical Shell with a Flat Head Closure under Internal Pressure. The forces and moments are shown in the positive sense above.

The following compatibility equations between cylinder and head are obtained from Reference 5, and are modified to conform to the notations of Fig. B.5.1.

$$\frac{M_o}{4pa^2} a_5 + \frac{Q_o}{2pa} a_6 + a_4 = \frac{M_o}{4pa^2} a_1 + \frac{Q_o}{2pa} a_2 + a_3 \tag{B.5.1}$$

$$\frac{M_o}{4pa^2} b_4 + \frac{Q_o}{2pa} b_5 = \frac{M_o}{4pa^2} b_1 + \frac{Q_o}{2pa} b_2 + b_3 \qquad (B.5.2)$$

where

$$a_1 = -\frac{6a}{h_2}(1-\nu) \qquad b_1 = \frac{6(1-\nu)}{\beta^2 h_1 h_2}$$

$$a_2 = 2(1-\nu) \qquad b_2 = -\frac{3(1-\nu)}{2\beta^2 a h_1}$$

$$a_3 = \frac{3a}{16h_2}(1-\nu) \qquad b_3 = -\frac{3(1-\nu)}{16\beta^2 h_1 h_2}$$

$$a_4 = -\frac{h_2}{h_1}\left(\frac{2-\nu}{8}\right) \qquad b_4 = -2\beta a\left(\frac{h_2}{h_1}\right)^2$$

$$a_5 = -2\frac{h_2}{h_1}\beta^2 a^2 \qquad b_5 = -\frac{1}{2}\left(\frac{h_2}{h_1}\right)^2$$

$$a_6 = -\frac{h_2}{h_1}\beta a \qquad \beta = \sqrt[4]{\frac{3(1-\nu^2)}{a^2 h_1^2}} \qquad (B.5.3)$$

Solution of Eqs. (B.5.1) and (B.5.2) yield the following expressions for Q_o and M_o. Specifically, they are:

$$M_o = 4pa^2\left[\frac{(a_4-a_3)(b_2-b_5)+b_3(a_2-a_6)}{(a_1-a_5)(b_2-b_5)-(a_2-a_6)(b_1-b_4)}\right] \qquad (B.5.4)$$

$$Q_o = 2pa\left[\frac{(a_3-a_4)(b_1-b_4)+b_3(a_5-a_1)}{(a_1-a_5)(b_2-b_5)-(a_2-a_6)(b_1-b_4)}\right] \qquad (B.5.5)$$

6. Short Circular Cylindrical Shell with Equal Thickness Flat Head Closures

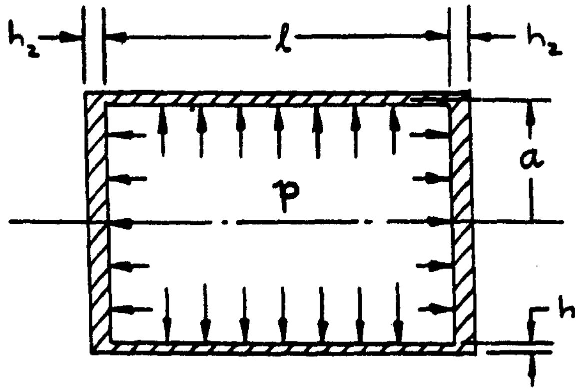


Fig. B.6.1 - Short Circular Cylindrical Shell with Equal Thickness Flat Head Closures under Internal Pressure

The signs convention and notations of Fig. B.6.1 will be used in the present analysis. For the case of a short cylindrical shell, the radial displacement (positive inward) δ, and the rotation $-\delta'$ at the junction due to the forces p, Q_o and M_o are, respectively,

$$\delta = -\frac{2Q_o\beta a^2}{Eh_1}\left(\frac{\cosh\beta\ell+\cos\beta\ell}{\sinh\beta\ell+\sin\beta\ell}\right)-\frac{2M_o\beta^2a^2}{Eh_1}\left(\frac{\sinh\beta\ell-\sin\beta\ell}{\sinh\beta\ell+\sin\beta\ell}\right)-\frac{pa^2}{2Eh_1}(2-\nu)$$

$$-\delta' = -\frac{2Q_o\beta^2a^2}{Eh_1}\left(\frac{\sinh\beta\ell-\sin\beta\ell}{\sinh\beta\ell+\sin\beta\ell}\right)-\frac{4M_o\beta^3a^2}{Eh_1}\left(\frac{\cosh\beta\ell-\cos\beta\ell}{\sinh\beta\ell+\sin\beta\ell}\right)$$

The dimensionless form of these two equations may be chosen to be

$$\frac{Eh_2\delta}{4pa^2} = \bar{a}_4 + \frac{M_0}{4pa^2}\bar{a}_5 + \frac{Q_0}{2pa}\bar{a}_6 \qquad \text{(B.6.1)}$$

$$-\frac{Eh_2^2\delta'}{8pa^3\beta^2 h_1} = \frac{M_0}{4pa^2}\bar{b}_4 + \frac{Q_0}{2pa}\bar{b}_5 \qquad \text{(B.6.2)}$$

where

$$\bar{a}_4 = -\frac{h_2}{h_1}\left(\frac{2-\nu}{8}\right)$$

$$\bar{a}_5 = -2\frac{h_2}{h_1}\beta^2 a^2\left(\frac{\sinh\beta\ell - \sin\beta\ell}{\sinh\beta\ell + \sin\beta\ell}\right)$$

$$\bar{a}_6 = -\frac{h_2}{h_1}\beta a\left(\frac{\cosh\beta\ell + \cos\beta\ell}{\sinh\beta\ell + \sin\beta\ell}\right)$$

$$\bar{b}_4 = -2\beta a\left(\frac{h_2}{h_1}\right)^2\left(\frac{\cosh\beta\ell - \cos\beta\ell}{\sinh\beta\ell + \sin\beta\ell}\right)$$

$$\bar{b}_5 = -\frac{1}{2}\left(\frac{h_2}{h_1}\right)^2\left(\frac{\sinh\beta\ell - \sin\beta\ell}{\sinh\beta\ell + \sin\beta\ell}\right) \qquad \text{(B.6.3)}$$

These "influence numbers" may be substituted in lieu of the influence numbers a_4, a_5, a_6, b_4 and b_5 for the long cylindrical shell in Eqs. (B.5.4) and (B.5.5) to give the expressions for the shear force and bending moment at the junction where the flat heads abutt the cylindrical shell, respectively.

Specifically, they are

$$M_0 = 4pa^2\left[\frac{(\bar{a}_4 - a_3)(b_2 - \bar{b}_5) + b_3(a_2 - \bar{a}_6)}{(a_1 - \bar{a}_5)(b_2 - \bar{b}_5) - (a_2 - \bar{a}_6)(b_1 - \bar{b}_4)}\right] \tag{B.6.4}$$

$$Q_0 = 2pa\left[\frac{(a_3 - \bar{a}_4)(b_1 - \bar{b}_4) + b_3(\bar{a}_5 - a_1)}{(a_1 - \bar{a}_5)(b_2 - \bar{b}_5) - (a_2 - \bar{a}_6)(b_1 - \bar{b}_4)}\right] \tag{B.6.5}$$

Defining

$$\begin{aligned}
\Omega_{14} &= \frac{\cosh\beta l + \cos\beta l}{\sinh\beta l + \sin\beta l} \\
\Omega_{15} &= \frac{\cosh\beta l - \cos\beta l}{\sinh\beta l + \sin\beta l}
\end{aligned} \tag{B.6.6}$$

and substituting Eqs. (B.4.4) and (B.5.3) into Eqs. (B.6.3), we obtain the following relationships:

$$\begin{aligned}
\bar{a}_4 &= a_4 \\
\bar{a}_5 &= a_5\Omega_{13} \\
\bar{a}_6 &= a_6\Omega_{14} \\
\bar{b}_4 &= b_4\Omega_{15} \\
\bar{b}_5 &= b_5\Omega_{13}
\end{aligned} \tag{B.6.7}$$

Equations (B.6.7) express the correspondence of the influence numbers between the long and short circular cylindrical shells. Substitution of these relationships into Eqs. (B.6.4) and (B.6.5) yields

$$M_o = 4pa^2 \left[\frac{(a_4 - a_3)(b_2 - b_5\Omega_8) + b_3(a_2 - a_6\Omega_{14})}{(a_1 - a_5\Omega_{13})(b_2 - b_5\Omega_{13}) - (a_2 - a_6\Omega_{14})(b_1 - b_4\Omega_{15})} \right] \tag{B.6.8}$$

$$Q_o = 2pa \left[\frac{(a_3 - a_4)(b_1 - b_4\Omega_{15}) + b_3(a_5\Omega_{13} - a_1)}{(a_1 - a_5\Omega_{13})(b_2 - b_5\Omega_{13}) - (a_2 - a_6\Omega_{14})(b_1 - b_4\Omega_{15})} \right] \tag{B.6.9}$$

7. Short Circular Cylindrical Shell with Unequal Thickness Flat Head Closures

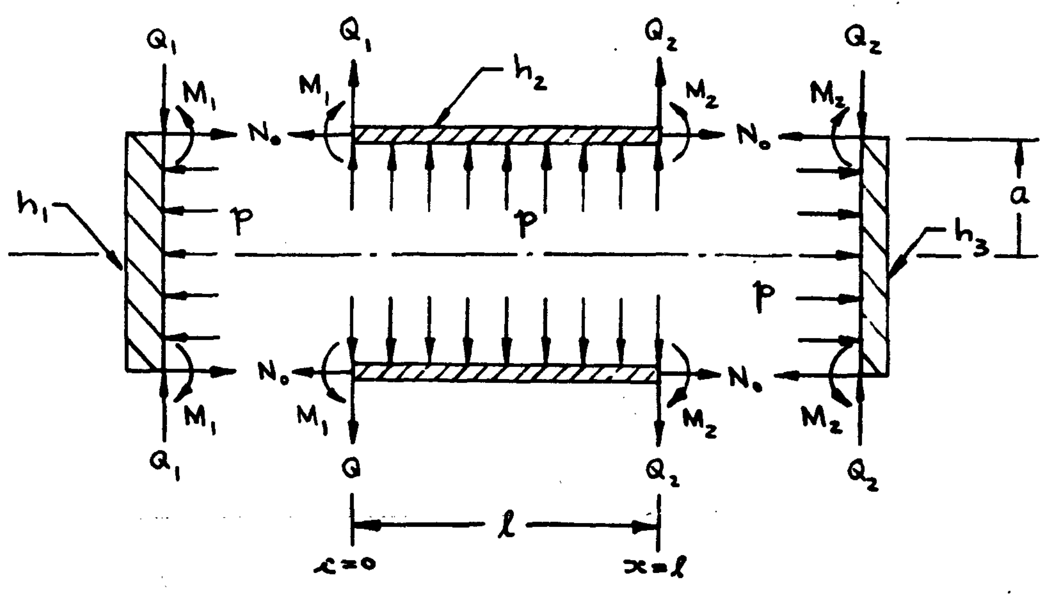


Fig. B.7.1 - Short Circular Cylindrical Shell with Unequal Thickness Flat Head Closures Under Internal Pressure

For the short cylindrical shell, the radial displacement (positive inward) δ, and the rotation $-\delta'$ at the junction where $x = 0$ due to the forces p, Q_1, Q_2, M_1, and M_2 are, respectively,

$$\delta_{x=0} = -\frac{Q_1(\sinh\beta l\cosh\beta l - \sin\beta l\cos\beta l) + Q_2(\cos\beta l\sinh\beta l - \sin\beta l\cosh\beta l)}{2\beta^3 D(\sinh^2\beta l - \sin^2\beta l)}$$

$$-\frac{M_1(\sinh^2\beta l + \sin^2\beta l) - 2M_2\sin\beta l\sinh\beta l}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)} - \frac{pa^2}{2Eh_2}(2-\nu) \qquad \text{(B.7.1)}$$

$$-\delta'_{x=0} = \frac{2Q_2\sin\beta l\sinh\beta l - Q_1(\sin^2\beta l + \sinh^2\beta l)}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)} -$$

$$\frac{M_1(\sin\beta l\cos\beta l + \sinh\beta l\cosh\beta l) - M_2(\sin\beta l\cosh\beta l + \cos\beta l\sinh\beta l)}{\beta D(\sinh^2\beta l - \sin^2\beta l)} \qquad \text{(B.7.2)}$$

For the junction $x = l$, the radial displacement and the rotation are, respectively,

$$\delta_{x=l} = -\frac{Q_1(\cos\beta l\sinh\beta l - \sin\beta l\cosh\beta l) + Q_2(\sinh\beta l\cosh\beta l - \sin\beta l\cos\beta l)}{2\beta^3 D(\sinh^2\beta l - \sin^2\beta l)}$$

$$+\frac{2\beta^2 a^2}{Eh_2}\left[\frac{2M_1\sin\beta l\sinh\beta l - M_2(\sinh^2\beta l + \sin^2\beta l)}{\sinh^2\beta l - \sin^2\beta l}\right] - \frac{pa^2}{2Eh_2}(2-\nu) \qquad \text{(B.7.3)}$$

$$-\delta'_{x=l} = -\frac{Q_2(\sin^2\beta l + \sinh^2\beta l) - 2Q_1\sin\beta l\sinh\beta l}{2\beta^2 D(\sinh^2\beta l - \sin^2\beta l)} +$$

$$\frac{4\beta^3 a^2}{Eh_2}\left[\frac{M_1(\sin\beta l\cosh\beta l + \cos\beta l\sinh\beta l) - M_2(\sin\beta l\cos\beta l + \sinh\beta l\cosh\beta l)}{\sinh^2\beta l - \sin^2\beta l}\right]$$

(B.7.4)

Equations (B.7.1), (B.7.2), (B.7.3), and (B.7.4) may be arranged in the following dimensionless forms:

$$\frac{Eh_1\delta_{x=0}}{4pa^2} = -\frac{h_1}{h_2}\left(\frac{2-\nu}{8}\right) + \frac{M_1}{4pa^2}\left[-2\frac{h_1}{h_2}\beta^2a^2\left(\frac{\sinh^2\beta\ell + \sin^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+ \frac{M_2}{4pa^2}\left[4\frac{h_1}{h_2}\beta^2a^2\left(\frac{\sin\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+ \frac{Q_1}{2pa}\left[-\frac{h_1}{h_2}\beta a\left(\frac{\sinh\beta\ell\cosh\beta\ell - \sin\beta\ell\cos\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+ \frac{Q_2}{2pa}\left[-\frac{h_1}{h_2}\beta a\left(\frac{\cos\beta\ell\sinh\beta\ell - \sin\beta\ell\cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right] \quad \text{(B.7.5)}$$

$$-\frac{Eh_1^2\delta'_{x=0}}{8\beta^2a^3h_2p} = \frac{M_1}{4pa^2}\left[-2\beta a\left(\frac{h_1}{h_2}\right)^2\left(\frac{\sin\beta\ell\cos\beta\ell + \sinh\beta\ell\cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+ \frac{M_2}{4pa^2}\left[2\beta a\left(\frac{h_1}{h_2}\right)^2\left(\frac{\sin\beta\ell\cosh\beta\ell + \cos\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+ \frac{Q_1}{2pa}\left[-\frac{1}{2}\left(\frac{h_1}{h_2}\right)^2\left(\frac{\sin^2\beta\ell + \sinh^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+ \frac{Q_2}{2pa}\left[\left(\frac{h_1}{h_2}\right)^2\left(\frac{\sin\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right] \quad \text{(B.7.6)}$$

$$\frac{Eh_3\delta_{x=\ell}}{4pa^2} = -\frac{h_3}{h_2}\left(\frac{2-\nu}{8}\right) + \frac{M_1}{4pa^2}\left[4\frac{h_3}{h_2}\beta^2a^2\left(\frac{\sin\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+\frac{M_2}{4pa^2}\left[-2\frac{h_3}{h_2}\beta^2a^2\left(\frac{\sinh^2\beta\ell + \sin^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+\frac{Q_1}{2pa}\left[-\frac{h_3}{h_2}\beta a\left(\frac{\cos\beta\ell\sinh\beta\ell - \sin\beta\ell\cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+\frac{Q_2}{2pa}\left[-\frac{h_3}{h_2}\beta a\left(\frac{\sinh\beta\ell\cosh\beta\ell - \sin\beta\ell\cos\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right] \quad \text{(B.7.7)}$$

$$-\frac{Eh_3^2\delta'_{x=\ell}}{8\beta^2a^3h_2p} = \frac{M_1}{4pa^2}\left[2\beta a\left(\frac{h_3}{h_2}\right)^2\left(\frac{\sin\beta\ell\cosh\beta\ell + \cos\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+\frac{M_2}{4pa}\left[-2\beta a\left(\frac{h_3}{h_2}\right)^2\left(\frac{\sin\beta\ell\cos\beta\ell + \sinh\beta\ell\cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+\frac{Q_1}{2pa}\left[\left(\frac{h_3}{h_2}\right)^2\left(\frac{\sin\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right]$$

$$+\frac{Q_2}{2pa}\left[-\frac{1}{2}\left(\frac{h_3}{h_2}\right)^2\left(\frac{\sin^2\beta\ell + \sinh^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}\right)\right] \quad \text{(B.7.8)}$$

Let us now define the following parameters:

$$\Omega_{20} = \frac{\sinh^2\beta\ell + \sin^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{21} = \frac{\sin\beta\ell\sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{16} = \frac{\sinh \beta\ell \cosh \beta\ell - \sin \beta\ell \cos \beta\ell}{\sinh^2 \beta\ell - \sin^2 \beta\ell}$$

$$\Omega_{18} = \frac{\cos \beta\ell \sinh \beta\ell - \sin \beta\ell \cosh \beta\ell}{\sinh^2 \beta\ell - \sin^2 \beta\ell}$$

$$\Omega_{17} = \frac{\sin \beta\ell \cos \beta\ell + \sinh \beta\ell \cosh \beta\ell}{\sinh^2 \beta\ell - \sin^2 \beta\ell}$$

$$\Omega_{19} = \frac{\sin \beta\ell \cosh \beta\ell + \cos \beta\ell \sinh \beta\ell}{\sinh^2 \beta\ell - \sin^2 \beta\ell} \tag{B.7.9}$$

Furthermore, let

$$A_1 = -\frac{h_1}{h_2}\left(\frac{2 - \nu}{8}\right)$$

$$A_2 = -2\frac{h_1}{h_2}\beta^2 a^2$$

$$A_3 = -\frac{h_1}{h_2}\beta a$$

$$A_4 = -\frac{h_3}{h_2}\left(\frac{2 - \nu}{8}\right)$$

$$A_5 = -2\frac{h_3}{h_2}\beta^2 a^2$$

$$A_6 = -\frac{h_3}{h_2}\beta a$$

$$A_7 = -2\beta a\left(\frac{h_1}{h_2}\right)^2$$

$$A_8 = -\frac{1}{2}\left(\frac{h_1}{h_2}\right)^2$$

(cont.)

$$A_9 = -2\beta a\left(\frac{h_3}{h_2}\right)^2$$

$$A_{10} = -\frac{1}{2}\left(\frac{h_3}{h_2}\right)^2 \qquad \text{(B.7.10)}$$

Equations (B.7.9) and (B.7.10) may now be substituted into Eqs. (B.7.5) through (B.7.8) to give the following set of dimensionless equations:

$$\frac{Eh_1\delta_{x=0}}{4pa^2} = A_1 + \frac{M_1}{4pa^2}(A_2\Omega_{20}) + \frac{M_2}{4pa^2}(-2A_2\Omega_{21}) + \frac{Q_1}{2pa}(A_3\Omega_{16}) + \frac{Q_2}{2pa}(A_3\Omega_{18}) \qquad \text{(B.7.11)}$$

$$\frac{Eh_3\delta_{x=l}}{4pa^2} = A_4 + \frac{M_1}{4pa^2}(-2A_5\Omega_{21}) + \frac{M_2}{4pa^2}(A_5\Omega_{20}) + \frac{Q_1}{2pa}(A_6\Omega_{18}) + \frac{Q_2}{2pa}(A_6\Omega_{16}) \qquad \text{(B.7.12)}$$

$$-\frac{Eh_1^2\delta'_{x=0}}{8\beta^2a^3h_2p} = \frac{M_1}{4pa^2}(A_7\Omega_{17}) + \frac{M_2}{4pa^2}(-A_7\Omega_{19}) + \frac{Q_1}{2pa}(A_8\Omega_{20}) + \frac{Q_2}{2pa}(-2A_8\Omega_{21}) \qquad \text{(B.7.13)}$$

$$-\frac{Eh_3^2\delta'_{x=l}}{8\beta^2a^3h_2p} = \frac{M_1}{4pa^2}(-A_9\Omega_{19}) + \frac{M_2}{4pa^2}(A_9\Omega_{17}) + \frac{Q_1}{2pa}(-2A_{10}\Omega_{21}) + \frac{Q_2}{2pa}(A_{10}\Omega_{20}) \qquad \text{(B.7.14)}$$

From Reference 5, we find the following dimensionless equations for the heads:

$$\frac{Eh_1\delta_{x=0}}{4pa^2} = \frac{M_1}{4pa^2}A_{11} + \frac{Q_1}{2pa}A_{12} + A_{13} \qquad \text{(B.7.15)}$$

$$\frac{Eh_3\delta_{x=\ell}}{4pa^2} = \frac{M_2}{4pa^2}A_{14} + \frac{Q_2}{2pa}A_{15} + A_{16} \qquad \text{(B.7.16)}$$

$$-\frac{Eh_1^2\delta'_{x=0}}{8\beta^2a^3h_2p} = \frac{M_1}{4pa^2}A_{17} + \frac{Q_1}{2pa}A_{18} + A_{19} \qquad \text{(B.7.17)}$$

$$-\frac{Eh_3^2\delta'_{x=\ell}}{8\beta^2a^3h_2p} = \frac{M_2}{4pa^2}A_{20} + \frac{Q_2}{2pa}A_{21} + A_{22} \qquad \text{(B.7.18)}$$

where

$$A_{11} = -\frac{6a}{h_1}(1-\nu)$$

$$A_{12} = A_{15} = 2(1-\nu)$$

$$A_{13} = \frac{3a}{16h_1}(1-\nu)$$

$$A_{14} = -\frac{6a}{h_3}(1-\nu)$$

$$A_{16} = \frac{3a}{16h_3}(1-\nu)$$

$$A_{17} = \frac{6(1-\nu)}{\beta^2h_1h_2}$$

$$A_{18} = -\frac{3(1-\nu)}{2\beta^2ah_2} = A_{21}$$

(cont.)

$$A_{19} = -\frac{3(1-\nu)}{16\beta^2 h_1 h_2}$$

$$A_{20} = \frac{6(1-\nu)}{\beta^2 h_2 h_3}$$

$$A_{22} = -\frac{3(1-\nu)}{16\beta^2 h_2 h_3} \qquad \text{(B.7.19)}$$

By equating Eqs. (B.7.11) with (B.7.15), (B.7.12) with (B.7.16), (B.7.13) with (B.7.17) and (B.7.14) with (B.7.18), we arrive at the following set of equations in M_1, M_2, Q_1, and Q_2. Specifically, they are:

$$\frac{M_1}{4pa^2}(A_2\Omega_{20} - A_{11}) + \frac{M_2}{4pa^2}(-2A_2\Omega_{21}) + \frac{Q_1}{2pa}(A_3\Omega_{16} - A_{12}) + \frac{Q_2}{2pa}(A_3\Omega_{18}) = A_{13} - A_1$$

$$\frac{M_1}{4pa^2}(-2A_5\Omega_{21}) + \frac{M_2}{4pa^2}(A_5\Omega_{20} - A_{14}) + \frac{Q_1}{2pa}(A_6\Omega_{18}) + \frac{Q_2}{2pa}(A_6\Omega_{16} - A_{15}) = A_{16} - A_4$$

$$\frac{M_1}{4pa^2}(A_7\Omega_{17} - A_{17}) + \frac{M_2}{4pa^2}(-A_7\Omega_{19}) + \frac{Q_1}{2pa}(A_8\Omega_{20} - A_{18}) + \frac{Q_2}{2pa}(-2A_8\Omega_{21}) = A_{19}$$

$$\frac{M_1}{4pa^2}(-A_9\Omega_{19}) + \frac{M_2}{4pa^2}(A_9\Omega_{17} - A_{20}) + \frac{Q_1}{2pa}(-2A_{10}\Omega_{21}) + \frac{Q_2}{2pa}(A_{10}\Omega_{20} - A_{21}) = A_{22}$$

(B.7.20)

Equation (B.7.20) may also be conveniently expressed in matrix form by defining

$$|U| = \begin{Bmatrix} M_1 \\ M_2 \\ Q_1 \\ Q_2 \end{Bmatrix} \; ; \qquad |V| = \begin{Bmatrix} A_{13} - A_1 \\ A_{16} - A_4 \\ A_{19} \\ A_{22} \end{Bmatrix} \; ; \qquad \text{(cont.)}$$

$$
[W] = \begin{bmatrix}
\frac{A_2\Omega_{20} - A_{11}}{4pa^2} & -\frac{2A_2\Omega_{21}}{4pa^2} & \frac{A_3\Omega_{16} - A_{12}}{2pa} & \frac{A_3\Omega_{18}}{2pa} \\
-\frac{2A_5\Omega_{21}}{4pa^2} & \frac{A_5\Omega_{20} - A_{14}}{4pa^2} & \frac{A_6\Omega_{18}}{2pa} & \frac{A_6\Omega_{16} - A_{15}}{2pa} \\
\frac{A_7\Omega_{17} - A_{17}}{4pa^2} & -\frac{A_7\Omega_{19}}{4pa^2} & \frac{A_8\Omega_{20} - A_{18}}{2pa} & -\frac{2A_8\Omega_{21}}{2pa} \\
-\frac{A_5\Omega_{19}}{4pa^2} & \frac{A_9\Omega_{17} - A_{20}}{4pa^2} & -\frac{2A_{10}\Omega_{21}}{2pa} & \frac{A_{10}\Omega_{20} - A_{21}}{2pa}
\end{bmatrix} \qquad \text{(B.7.21)}
$$

Then,

$$
|W|\,|U| = |V|
$$

and

$$
|U| = |W|^{-1}|V| \qquad \text{(B.7.22)}
$$

8. Long Circular Cylindrical Shell with an Ellipsoidal Head Closure

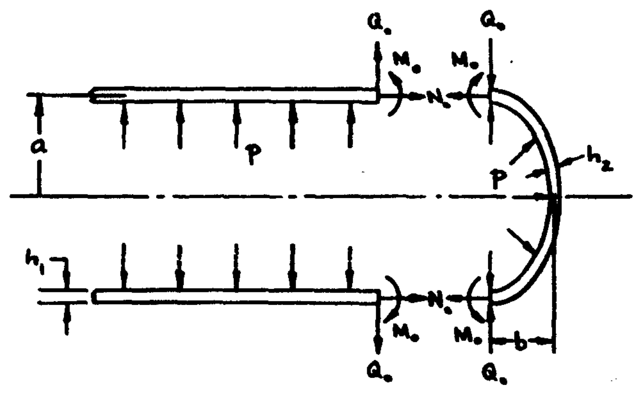


Fig. B.8.1 - Forces and Moments at the Junction of the Long Circular Cylindrical Shell and the Ellipsoidal Head Closure under Internal Pressure

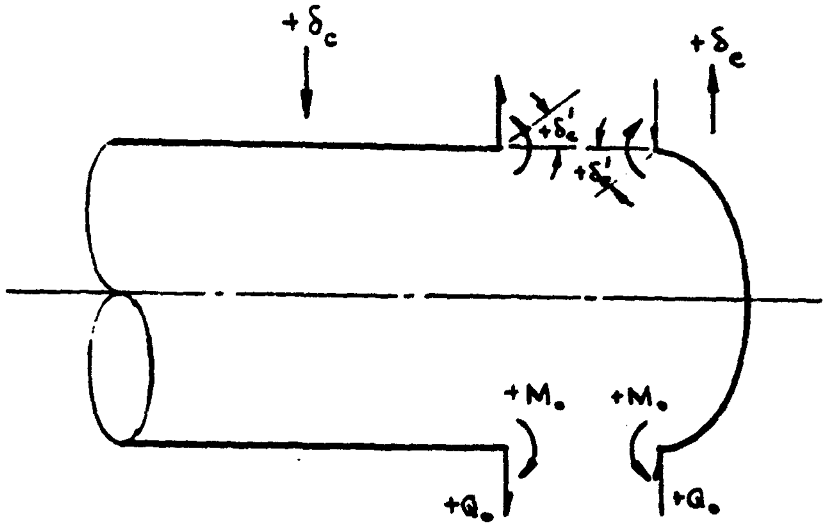


Fig. B.8.2 - Signs Convention for Middle Surface Radial Deflection and Rotation at the Junction

The radial deflection and the rotation of the long circular cylindrical shell at the junction produced by the loads p, Q_o, and M_o are, respectively

$$\delta_c = -\frac{1}{2\beta_c^3 D_c}\left(\beta_c M_o + Q_o\right) - \frac{pa^2}{2Eh_1}(2-\nu) \tag{B.8.1}$$

$$\delta_c' = \frac{1}{2\beta_c^2 D_c}\left(2\beta_c M_o + Q_o\right) \tag{B.8.2}$$

where

$$\begin{aligned} \beta_c^4 &= \frac{Eh_1}{4a^2 D_c} = \frac{3(1-\nu^2)}{a^2 h_1^2} \\ D_c &= \frac{Eh_1^3}{12(1-\nu^2)} \end{aligned} \tag{B.8.3}$$

The radial deflection and the rotation of the ellipsoidal head at the junction produced by the loads p, Q_o, and M_o are obtained from Reference 6, and are, respectively,

$$\delta_e = \frac{1}{2\beta_{e_o}^3 D_e}\left(-Q_o + \beta_{e_o} M_o\right) + \frac{pa^2}{2Eh_2}\left(2 - \frac{a^2}{b^2} - \nu\right) \quad \text{(B.8.4)}$$

$$\delta_e' = \frac{1}{2\beta_{e_o}^2 D_e}\left(Q_o - 2\beta_{e_o} M_o\right) \quad \text{(B.8.5)}$$

where

$$\beta_{e_o}^4 = \frac{Eh_2}{4a^2 D_e} = \frac{3(1-\nu^2)}{a^2 h_2^2}$$

$$D_e = \frac{Eh_2^3}{12(1-\nu^2)} \quad \text{(B.8.6)}$$

For strain compatibility, we must have

$$\delta_c = -\delta_e$$

$$\delta_c' = \delta_e' \quad \text{(B.8.7)}$$

Hence,

$$-\frac{1}{2\beta_c^3 D_c}\left(\beta_c M_o + Q_o\right) - \frac{pa^2}{2Eh_1}(2-\nu)$$
$$= -\frac{1}{2\beta_{e_o}^3 D_e}\left(-Q_o + \beta_{e_o} M_o\right) - \frac{pa^2}{2Eh_2}\left(2 - \frac{a^2}{b^2} - \nu\right)$$

$$\frac{1}{2\beta_c^2 D_c}\left(2\beta_c M_o + Q_o\right) = \frac{1}{2\beta_{e_o}^2 D_e}\left(Q_o - 2\beta_{e_o} M_o\right)$$

from which

$$\left(\frac{1}{\beta_{e_o}^2 D_e} - \frac{1}{\beta_c^2 D_c}\right) M_o - \left(\frac{1}{\beta_c^3 D_c} + \frac{1}{\beta_{e_o}^3 D_e}\right) Q_o = \frac{pa^2}{Eh_2}\left[(2-\nu)\left(\frac{h_2}{h_1} - 1\right) + \frac{a^2}{b^2}\right] \tag{B.8.8}$$

$$\left(\frac{1}{\beta_{e_o} D_e} + \frac{1}{\beta_c D_c}\right) M_o + \frac{1}{2}\left(\frac{1}{\beta_c^2 D_c} - \frac{1}{\beta_{e_o}^2 D_e}\right) Q_o = 0 \tag{B.8.9}$$

Simultaneous solution of Eqs. (B.8.8) and (B.8.9) yields

$$M_o = \frac{\frac{pa^2}{2Eh_2}\left[(2-\nu)\left(\frac{h_2}{h_1} - 1\right) + \frac{a^2}{b^2}\right]\left(\frac{1}{\beta_c^2 D_c} - \frac{1}{\beta_{e_o}^2 D_e}\right)}{\frac{1}{2}\left(\frac{1}{\beta_c^2 D_c} - \frac{1}{\beta_{e_o}^2 D_e}\right)\left(\frac{1}{\beta_{e_o}^2 D_e} - \frac{1}{\beta_c^2 D_c}\right) + \left(\frac{1}{\beta_{e_o} D_e} + \frac{1}{\beta_c D_c}\right)\left(\frac{1}{\beta_c^3 D_c} + \frac{1}{\beta_{e_o}^3 D_e}\right)} \tag{B.8.10}$$

$$Q_o = \frac{-\frac{pa^2}{Eh_2}\left[(2-\nu)\left(\frac{h_2}{h_1} - 1\right) + \frac{a^2}{b^2}\right]\left(\frac{1}{\beta_{e_o} D_e} + \frac{1}{\beta_c D_c}\right)}{\frac{1}{2}\left(\frac{1}{\beta_c^2 D_c} - \frac{1}{\beta_{e_o}^2 D_e}\right)\left(\frac{1}{\beta_{e_o}^2 D_e} - \frac{1}{\beta_c^2 D_c}\right) + \left(\frac{1}{\beta_{e_o} D_e} + \frac{1}{\beta_c D_c}\right)\left(\frac{1}{\beta_c^3 D_c} + \frac{1}{\beta_{e_o}^3 D_e}\right)} \tag{B.8.11}$$

These formulas for M_0 and Q_0 may be put into the following dimensionless forms:

$$\frac{M_0}{\dfrac{pa^2\beta_c^2 D_c}{2Eh_2}\left[(2-\nu)\left(\dfrac{h_2}{h_1}-1\right)+\dfrac{a^2}{b^2}\right]} = \frac{1-\left(\dfrac{h_1}{h_2}\right)^2}{\left[1+\left(\dfrac{h_1}{h_2}\right)^{5/2}\right]\left[1+\left(\dfrac{h_1}{h_2}\right)^{3/2}\right]-\dfrac{1}{2}\left[1-\left(\dfrac{h_1}{h_2}\right)^2\right]^2} \quad \text{(B.8.12)}$$

$$\frac{Q_0}{\dfrac{pa^2\beta_c^3 D_c}{Eh_2}\left[(2-\nu)\left(\dfrac{h_2}{h_1}-1\right)+\dfrac{a^2}{b^2}\right]} = \frac{1+\left(\dfrac{h_1}{h_2}\right)^{5/2}}{\left[1+\left(\dfrac{h_1}{h_2}\right)^{5/2}\right]\left[1+\left(\dfrac{h_1}{h_2}\right)^{3/2}\right]-\dfrac{1}{2}\left[1-\left(\dfrac{h_1}{h_2}\right)^2\right]^2} \quad \text{(B.8.13)}$$

9. <u>Short Circular Cylindrical Shell with Equal Thickness Ellipsoidal Head Closures</u>

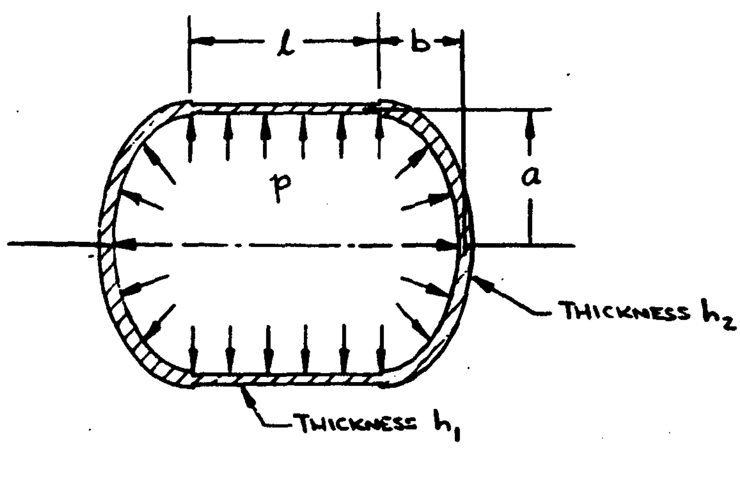


Fig. B.9.1 - Short Circular Cylindrical Shell with Equal Thickness Ellipsoidal Head Closures under Internal Pressure

The signs convention and notations of Fig. B.8.2 will be used in the present analysis. For the short circular cylindrical shell, the dimensionless forms of the equations for radial displacement δ_c and the rotation δ_c' at the junction due to the forces p, Q_0, and M_0 are given by Eqs. (B.6.1) and (B.6.2), respectively. These are listed in the following forms:

$$\frac{Eh_2\delta_c}{4pa^2} = a_4 + \frac{M_0}{4pa^2} a_5\Omega_{13} + \frac{Q_0}{2pa} a_6\Omega_{14} \quad \text{(B.9.1)}$$

$$-\frac{Eh_2^2\delta_c'}{8pa^3\beta_c^2 h_1} = \frac{M_0}{4pa^2} b_4\Omega_{15} + \frac{Q_0}{2pa} b_5\Omega_{13} \quad \text{(B.9.2)}$$

where a_4, a_5, a_6, b_4 and b_5 are as defined in Eq. (B.5.3); Ω_{13} by Eq. (B.4.4) and Ω_{14} and Ω_{15} by Eq. (B.6.6).

For the ellipsoidal head closure, the radial displacement δ_e and the rotation δ_e' at the junction due to the forces p, Q_0, and M_0 are given by Eqs. (B.8.4) and (B.8.5). These two equations may be written in the following dimensionless forms:

$$-\frac{Eh_2\delta_e}{4pa^2} = \frac{M_0}{4pa^2} a_1 + \frac{Q_0}{2pa} a_2 + a_3 \quad \text{(B.9.3)}$$

$$-\frac{Eh_2^2\delta_e'}{8pa^3\beta_e^2 h_1} = \frac{M_0}{4pa^2} b_1 + \frac{Q_0}{2pa} b_2 \quad \text{(B.9.4)}$$

where

$$a_1 = -\frac{2a}{h_2}\sqrt{3(1-\nu^2)}$$

$$a_2 = \frac{1}{\beta_e h_2}\sqrt{3(1-\nu^2)}$$

(cont.)

$$a_3 = -\frac{1}{8}\left(2 - \frac{a^2}{b^2} - \nu\right)$$

$$b_1 = \frac{2\beta_{e_0}}{\beta_c^2 h_1}\sqrt{3(1-\nu^2)}$$

$$b_2 = -\frac{\sqrt{3(1-\nu^2)}}{2\beta_c^2 a h_1} \qquad \text{(B.9.5)}$$

In accordance with the strain compatibility equations (B.8.7), we have the following equalities:

$$a_4 + \frac{M_0}{4pa^2} a_5\Omega_{13} + \frac{Q_0}{2pa} a_6\Omega_{14} = \frac{M_0}{4pa^2} a_1 + \frac{Q_0}{2pa} a_2 + a_3$$

$$\frac{M_0}{4pa^2} b_4\Omega_{15} + \frac{Q_0}{2pa} b_5\Omega_{13} = \frac{M_0}{4pa^2} b_1 + \frac{Q_0}{2pa} b_2$$

from which

$$M_0 = 4pa^2\left[\frac{(a_4 - a_3)(b_2 - b_5\Omega_{13})}{(a_1 - a_5\Omega_{13})(b_2 - b_5\Omega_{13}) - (a_2 - a_6\Omega_{14})(b_1 - b_4\Omega_{15})}\right] \qquad \text{(B.9.6)}$$

$$Q_0 = 2pa\left[\frac{(a_3 - a_4)(b_1 - b_4\Omega_{15})}{(a_1 - a_5\Omega_{13})(b_2 - b_5\Omega_{13}) - (a_2 - a_6\Omega_{14})(b_1 - b_4\Omega_{15})}\right] \qquad \text{(B.9.7)}$$

10. Short Circular Cylindrical Shell with Unequal Thickness Ellipsoidal Head Closures

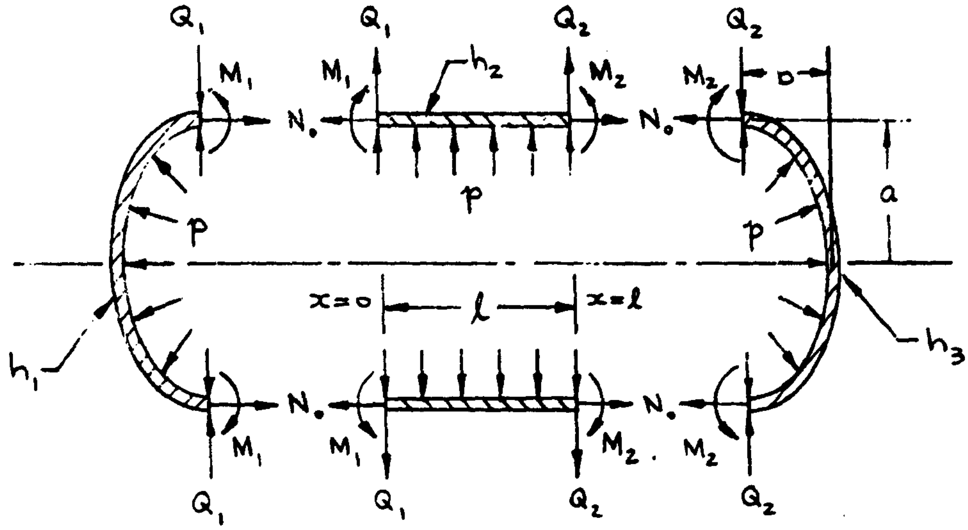


Fig. B. 10.1 - Short Circular Cylindrical Shell with Unequal Thickness Ellipsoidal Head Closures under Internal Pressure

The signs convention of Fig. B. 8. 2 will be adhered to in this analysis. For the short circular cylindrical shell, the radial displacements and rotations at the junctions where $x = 0$ and $x = \ell$, respectively, due to the forces p, Q_1, Q_2, M_1, and M_2 are given by Eqs. (B. 7. 11) through (B. 7. 14) in dimensionless forms and will not be repeated here.

For the ellipsoidal heads, the corresponding dimensionless forms of the radial displacements and rotations are:

$$-\frac{E h_1 \delta_{x=0}}{4pa^2} = \frac{M_1}{4pa^2} A_{11} + \frac{Q_1}{2pa} A_{12} + A_{13} \qquad \text{(B. 10. 1)}$$

$$-\frac{E h_3 \delta_{x=\ell}}{4pa^2} = \frac{M_2}{4pa^2} A_{14} + \frac{Q_2}{2pa} A_{15} + A_{16} \qquad \text{(B. 10. 2)}$$

$$-\frac{Eh_1^2 \delta'_{x=0}}{8\beta_c^2 a^3 h_2 p} = \frac{M_1}{4pa^2} A_{17} + \frac{Q_1}{2pa} A_{18} \qquad \text{(B. 10. 3)}$$

$$-\frac{Eh_3^2 \delta'_{x=\ell}}{8\beta_0^2 a^3 h_2 p} = \frac{M_2}{4pa^2} A_{19} + \frac{Q_2}{2pa} A_{20} \qquad \text{(B. 10. 4)}$$

where

$$A_{11} = -\frac{2a}{h_1}\sqrt{3(1-\nu^2)}$$

$$A_{12} = \frac{1}{\beta_{10} h_1}\sqrt{3(1-\nu^2)}$$

$$A_{13} = A_{16} = -\frac{1}{8}\left(2-\frac{a^2}{b^2}-\nu\right)$$

$$A_{14} = -\frac{2a}{h_3}\sqrt{3(1-\nu^2)}$$

$$A_{15} = \frac{1}{\beta_{30} h_3}\sqrt{3(1-\nu^2)}$$

$$A_{17} = \frac{2\beta_{10}}{\beta_c^2 h_2}\sqrt{3(1-\nu^2)}$$

$$A_{18} = -\frac{\sqrt{3(1-\nu^2)}}{2\beta_c^2 a h_2} = A_{20}$$

$$A_{19} = \frac{2\beta_{30}}{\beta_c^2 h_2}\sqrt{3(1-\nu^2)}$$

$$\beta_{10}^4 = \frac{Eh_1}{4a^2 D_1} = \frac{3(1-\nu^2)}{a^2 h_1^2}$$

$$\beta_{30}^4 = \frac{Eh_3}{4a^2 D_3} = \frac{3(1-\nu^2)}{a^2 h_3^2}$$

(cont.)

$$D_1 = \frac{Eh_1^3}{12(1-\nu^2)}$$

$$D_3 = \frac{Eh_3^3}{12(1-\nu^2)} \qquad \text{(B.10.5)}$$

Equating now Eqs. (B.7.11) with (B.10.1), (B.7.12) with (B.10.2), (B.7.13) with (B.10.3), and (B.7.14) with (B.10.4) in accordance with the strain compatibility equations (B.8.7), we arrive at the following set of equations in M_1, M_2, Q_1, and Q_2. Specifically, they are:

$$\frac{M_1}{4pa^2}(A_2\Omega_{20}-A_{11}) + \frac{M_2}{4pa^2}(-2A_2\Omega_{21}) + \frac{Q_1}{2pa}(A_3\Omega_{16}-A_{12}) + \frac{Q_2}{2pa}(A_3\Omega_{18}) = A_{13}-A_1$$

$$\frac{M_1}{4pa^2}(-2A_5\Omega_{21}) + \frac{M_2}{4pa^2}(A_5\Omega_{20}-A_{14}) + \frac{Q_1}{2pa}(A_6\Omega_{18}) + \frac{Q_2}{2pa}(A_6\Omega_{16}-A_{15}) = A_{16}-A_4$$

$$\frac{M_1}{4pa^2}(A_7\Omega_{17}-A_{17}) + \frac{M_2}{4pa^2}(-A_7\Omega_{19}) + \frac{Q_1}{2pa}(A_8\Omega_{20}-A_{18}) + \frac{Q_2}{2pa}(-2A_8\Omega_{21}) = 0$$

$$\frac{M_1}{4pa^2}(-A_9\Omega_{19}) + \frac{M_2}{4pa^2}(A_9\Omega_{17}-A_{19}) + \frac{Q_1}{2pa}(-2A_{10}\Omega_{21}) + \frac{Q_2}{2pa}(A_{10}\Omega_{20}-A_{20}) = 0$$

Let us now define the following matrices:

$$\{U\} = \begin{Bmatrix} M_1 \\ M_2 \\ Q_1 \\ Q_2 \end{Bmatrix}; \qquad \{V\} = \begin{Bmatrix} A_{13}-A_1 \\ A_{16}-A_4 \\ 0 \\ 0 \end{Bmatrix};$$

(cont.)

$$
[W] = \begin{bmatrix}
\frac{A_2\Omega_{20} - A_{11}}{4pa^2} & -\frac{2A_2\Omega_{21}}{4pa^2} & \frac{A_3\Omega_{16} - A_{12}}{2pa} & \frac{A_3\Omega_{18}}{2pa} \\
-\frac{2A_5\Omega_{21}}{4pa^2} & \frac{A_5\Omega_{20} - A_{14}}{4pa^2} & \frac{A_6\Omega_{18}}{2pa} & \frac{A_6\Omega_{16} - A_{15}}{2pa} \\
\frac{A_7\Omega_{17} - A_{17}}{4pa^2} & -\frac{A_7\Omega_{19}}{4pa^2} & \frac{A_8\Omega_{20} - A_{18}}{2pa} & -\frac{2A_8\Omega_{21}}{2pa} \\
-\frac{A_9\Omega_{19}}{4pa^2} & \frac{A_9\Omega_{17} - A_{19}}{4pa^2} & -\frac{2A_{10}\Omega_{21}}{2pa} & \frac{A_{10}\Omega_{20} - A_{20}}{2pa}
\end{bmatrix} \qquad (B.10.6)
$$

Then the four simultaneous equations listed above may be written in the form

$$
[W] \; \{U\} = \{V\}
$$

from which

$$
\{U\} = [W]^{-1} \; \{V\} \qquad (B.10.7)
$$

The coefficients A_1 through A_{10} in the above equations are defined in Eq. (B.7.10) where it is understood that $\beta = \beta_c$. The functions Ω_{20} through Ω_{19} are identically defined in Eq. (B.7.9).

11. Long Circular Cylindrical Shell with a Hemispherical Head Closure

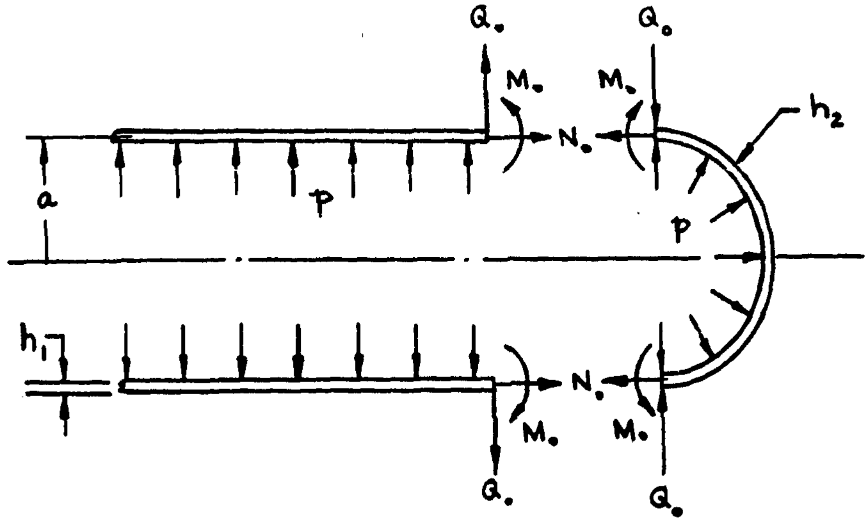


Fig. B.11.1 - Forces and Moments at the Junction of the Long Circular Cylindrical Shell and the Hemispherical Head Closure under Internal Pressure

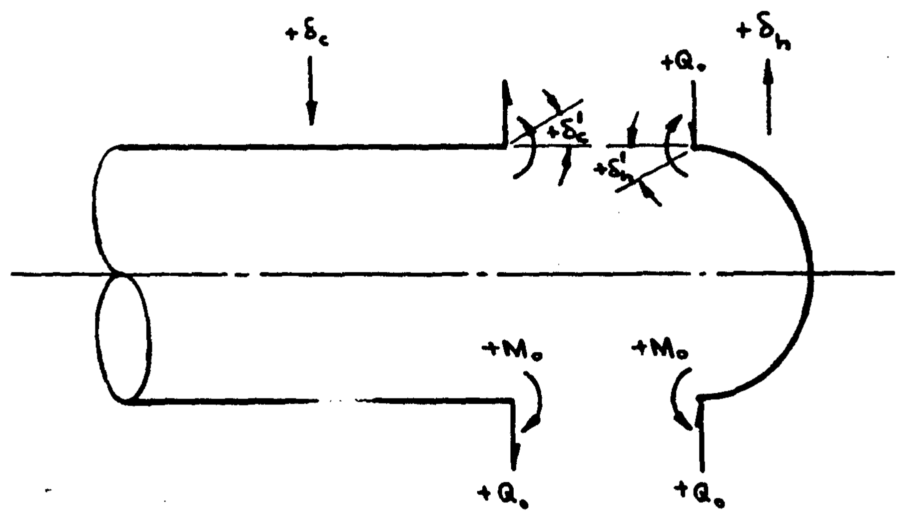


Fig. B.11.2 - Signs Convention for Middle Surface Radial Deflection and Rotation at the Junction

The discontinuity forces Q_0 and moments M_0 at the junction of the long circular cylindrical shell and the hemispherical head closure under the action of internal pressure can be readily obtained from Eqs. (B.8.12) and (B.8.13) developed for the case with the ellipsoidal head configuration; it being necessary to replace the quantity a^2/b^2 by unity. In so doing, we find

$$\frac{M_0}{\frac{pa^2\beta_c^2 D_c}{2Eh_2}\left[\frac{h_2}{h_1}(2-\nu)-(1-\nu)\right]} = \frac{1-\left(\frac{h_1}{h_2}\right)^2}{\left[1+\left(\frac{h_1}{h_2}\right)^{5/2}\right]\left[1+\left(\frac{h_1}{h_2}\right)^{3/2}\right]-\frac{1}{2}\left[1-\left(\frac{h_1}{h_2}\right)^2\right]^2} \tag{B.11.1}$$

$$\frac{Q_0}{\frac{pa^2\beta_c^3 D_c}{Eh_2}\left[\frac{h_2}{h_1}(2-\nu)-(1-\nu)\right]} = \frac{1+\left(\frac{h_1}{h_2}\right)^{5/2}}{\left[1+\left(\frac{h_1}{h_2}\right)^{5/2}\right]\left[1+\left(\frac{h_1}{h_2}\right)^{3/2}\right]-\frac{1}{2}\left[1-\left(\frac{h_1}{h_2}\right)^2\right]^2} \tag{B.11.2}$$

12. Short Circular Cylindrical Shell with Equal Thickness Hemispherical Head Closures

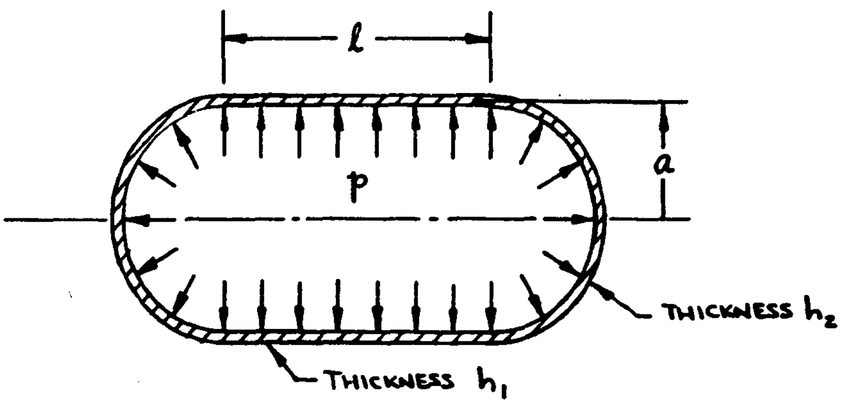


Fig. B.12.1 - Short Circular Cylindrical Shell with Equal Thickness Hemispherical Head Closures under Internal Pressure

The formulas developed for the case of the short circular cylindrical shell with equal thickness ellipsoidal head closures are readily adaptable to the present configuration by substituting unity in place of the quantity a^2/b^2. For example, the quantity a_3 in Eq. (B.9.6) for the bending moment M_0 and in Eq. (B.9.7) for the shear force Q_0 must now be replaced by

$$a_3 = -\frac{1}{8}(1-\nu) \tag{B.12.1}$$

13. Short Circular Cylindrical Shell with Unequal Thickness Hemispherical Head Closures

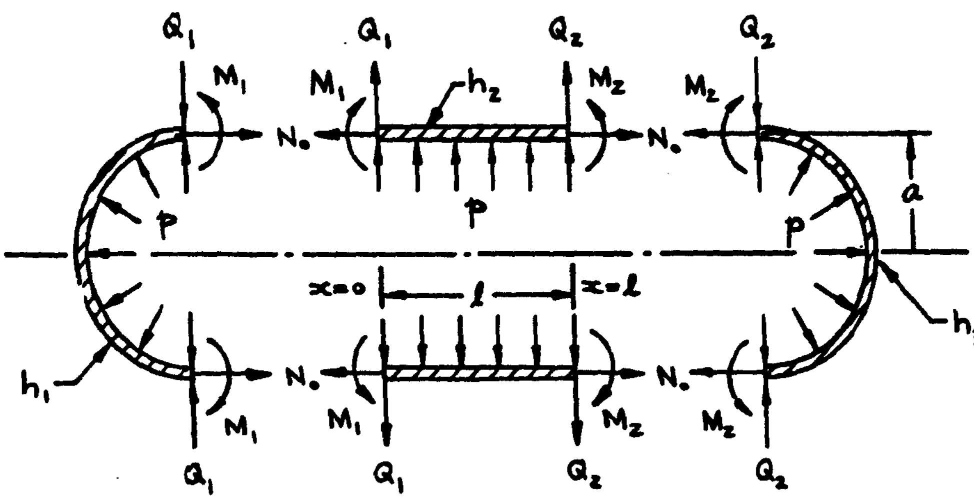


Fig. B.13.1 - Short Circular Cylindrical Shell with Unequal Thickness Hemispherical Head Closures under Internal Pressure

Formulas (B.10.6) and (B.10.7) for the determination of the bending moments M_1 and M_2, and the shear forces Q_1 and Q_2 developed for the case of a short circular cylindrical shell with unequal thickness ellipsoidal head closures are also applicable to the present configuration with the exception that the quantities A_{13} and A_{16} defined in Eq. (B.10.5) must now be replaced by

$$A_{13} = A_{16} = -\frac{1}{8}(1-\nu) \tag{B.13.1}$$

14. Long Circular Cylindrical Shell with a Conical Head Closure

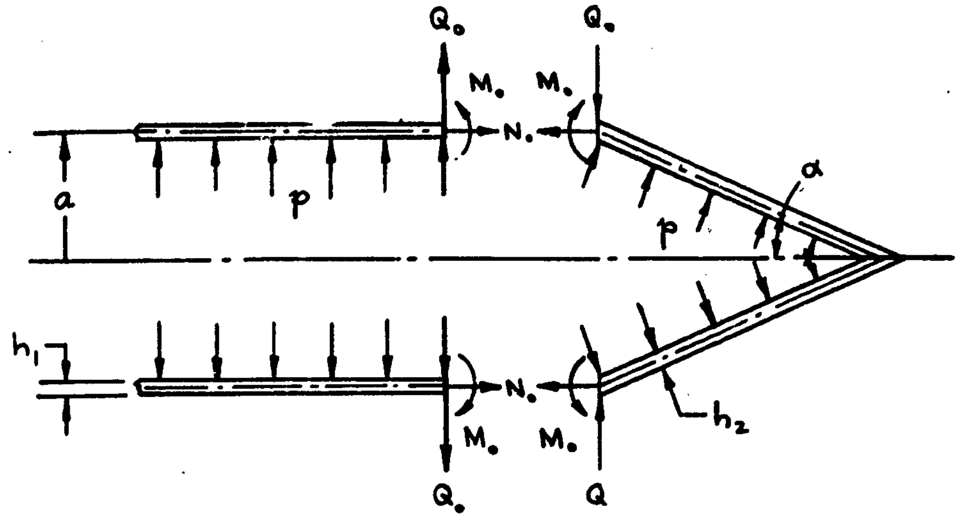


Fig. B. 14. 1 - Forces and Moments at the Junction of the Long Circular Cylindrical Shell and the Conical Head Closure under Internal Pressure

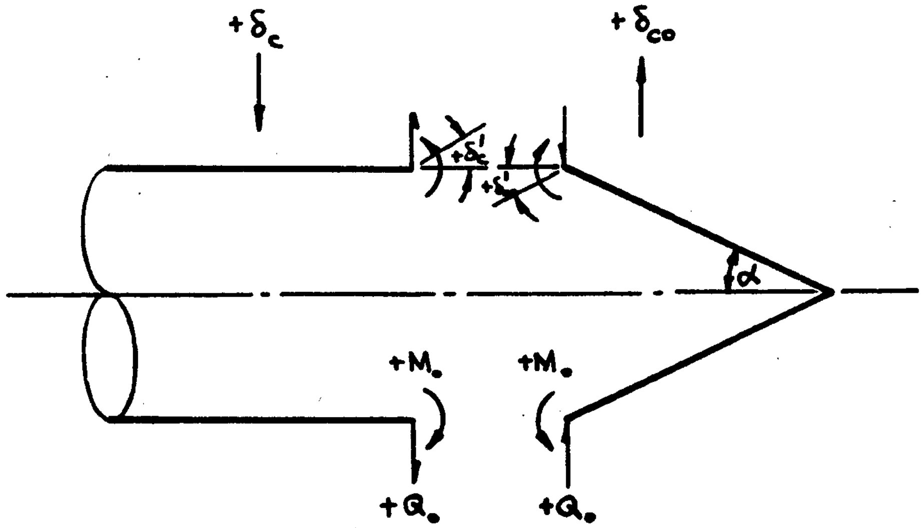


Fig. B. 14. 2 - Signs Convention for Middle Surface Radial Deflection and Rotation at the Junction

The radial deflection and the rotation of the long circular cylindrical shell at the junction produced by the loads p, Q_0, and M_0 in accordance with the signs convention of Fig. B.14.2 are, respectively,

$$\delta_c = -\frac{1}{2\beta_c^3 D_c}(\beta_c M_o + Q_o) - \frac{pa^2}{2Eh_1}(2-\nu) \qquad \text{(B. 14. 1)}$$

$$\delta_c' = \frac{1}{2\beta_c^2 D_c}(2\beta_c M_o + Q_o) \qquad \text{(B. 14. 2)}$$

where

$$\beta_c^4 = \frac{Eh_1}{4a^2 D_c} = \frac{3(1-\nu^2)}{a^2 h_1^2}$$
$$D_c = \frac{Eh_1^3}{12(1-\nu^2)} \qquad \text{(B. 14. 3)}$$

The dimensionless forms of Eqs. (B. 14. 1) and (B. 14. 2) may be chosen to be

$$\frac{Eh_2\delta_c}{4pa^2} = \frac{M_o}{4pa^2}a_4 + \frac{Q_o}{2pa}a_5 + a_6 \qquad \text{(B. 14. 4)}$$

$$\frac{Eh_2^2\delta_c'}{8pa^3\beta_c^2 h_1} = \frac{M_o}{4pa^2}b_4 + \frac{Q_o}{2pa}b_5 \qquad \text{(B. 14. 5)}$$

where

$$a_4 = -2\frac{h_2}{h_1}\beta_c^2 a^2$$

$$a_5 = -\frac{h_2}{h_1}\beta_c a$$

$$a_6 = -\frac{h_2}{h_1}\left(\frac{2-\nu}{8}\right)$$

$$b_4 = 2\beta_c a\left(\frac{h_2}{h_1}\right)^2$$

$$b_5 = \frac{1}{2}\left(\frac{h_2}{h_1}\right)^2 \qquad \text{(B. 14. 6)}$$

For the conical head, the corresponding dimensionless forms are:

$$\frac{Eh_2\delta_{co}}{4pa^2} = \frac{M_o}{4pa^2}a_1 + \frac{Q_o}{2pa}a_2 + a_3 \qquad \text{(B. 14. 7)}$$

$$\frac{Eh_2^2\delta'_{co}}{8pa^3\beta_c^2h_1} = \frac{M_o}{4pa^2}b_1 + \frac{Q_o}{2pa}b_2 + b_3 \qquad \text{(B. 14. 8)}$$

The quantities a_1, a_2, a_3, b_1, b_2, and b_3 are dimensionless "influence numbers" for the conical head and can be obtained from either References 2 or 3. In the notations of Fig. B. 14. 1 and the signs convention of Fig. B. 14.2, these influence numbers become

$$a_1 = -\frac{b_2\,\mathfrak{z}_o^2\tan\alpha\sin\alpha}{2}$$

(cont.)

$$a_2 = -\frac{\frac{\xi_o^2 B}{4} - \nu^2 G}{C + 2\nu G} \sin\alpha$$

$$a_3 = \frac{2-\nu}{8}\sec\alpha + \frac{a_2}{4}\tan\alpha - \frac{3b_2(1+\nu)}{\xi_o^2}\sec\alpha$$

$$b_1 = -\frac{2G}{C+2\nu G}\frac{\xi_o^2 \tan\alpha}{4}$$

$$b_2 = -\frac{A/2}{C+2\nu G}$$

$$b_3 = -\frac{6(1+\nu)}{\xi_o^4} b_1 \operatorname{cosec}^2\alpha + \frac{b_2\tan\alpha}{4} - \frac{3}{2}\frac{\sec\alpha \operatorname{cosec}\alpha}{\xi_o^2} \qquad \text{(B. 14. 9)}$$

where

$$A = \xi_o(\mathrm{ber}_2'\xi_o \,\mathrm{bei}_2\xi_o - \mathrm{bei}_2'\xi_o\,\mathrm{ber}_2\xi_o)$$

$$B = (\mathrm{ber}_2'\xi_o)^2 + (\mathrm{bei}_2'\xi_o)^2$$

$$C = \xi_o(\mathrm{ber}_2\xi_o\,\mathrm{ber}_2'\xi_o + \mathrm{bei}_2\xi_o\,\mathrm{bei}_2'\xi_o)$$

$$G = (\mathrm{ber}_2\xi_o)^2 + (\mathrm{bei}_2\xi_o)^2$$

$$\xi_o = 2\sqrt[4]{12(1-\nu^2)}\sqrt{\frac{a}{h_2}\cot\alpha\operatorname{cosec}\alpha} \qquad \text{(B. 14. 10)}$$

For strain compatibility, we must have

$$\delta_c = -\delta_{co}$$

$$\delta_c' = \delta_{co}' \qquad \text{(B.14.11)}$$

Hence,

$$\frac{M_o}{4pa^2}a_4+\frac{Q_o}{2pa}a_5+a_6 = -\frac{M_o}{4pa^2}a_1-\frac{Q_o}{2pa}a_2-a_3$$
$$\frac{M_o}{4pa^2}b_4+\frac{Q_o}{2pa}b_5 = \frac{M_o}{4pa^2}b_1+\frac{Q_o}{2pa}b_2+b_3 \qquad \text{(B. 14. 12)}$$

or

$$\frac{M_o}{4pa^2}(a_1+a_4)+\frac{Q_o}{2pa}(a_2+a_5) = -(a_3+a_6)$$
$$\frac{M_o}{4pa^2}(b_4-b_1)+\frac{Q_o}{2pa}(b_5-b_2) = b_3 \qquad \text{(B. 14. 13)}$$

from which

$$M_o = 4pa^2\left[\frac{(a_3+a_6)(b_2-b_5)-b_3(a_2+a_5)}{(a_1+a_4)(b_5-b_2)+(b_1-b_4)(a_2+a_5)}\right] \qquad \text{(B. 14. 14)}$$

$$Q_o = 2pa\left[\frac{b_3(a_1+a_4)+(a_3+a_6)(b_4-b_1)}{(a_1+a_4)(b_5-b_2)+(b_1-b_4)(a_2+a_5)}\right] \qquad \text{(B. 14. 15)}$$

The Bessel-Kelvin functions ber, bei, ker, kei, and their first derivatives have been extensively tabulated by Lowell (Ref. 7). Higher order Bessel-Kelvin functions are related to the aforementioned functions by the following recurrent formulas (Ref. 8):

$$\mathrm{ber}_1\,\xi = \frac{1}{\sqrt{2}}\left(\mathrm{ber}'\xi-\mathrm{bei}'\xi\right) \qquad \text{(cont.)}$$

$$\mathrm{bei}_1\,\xi = \frac{1}{\sqrt{2}}\left(\mathrm{ber}'\xi + \mathrm{bei}'\xi\right)$$

$$\mathrm{ber}_{n+1}\,\xi = -\frac{n\sqrt{2}}{\xi}\left(\mathrm{ber}_n\,\xi - \mathrm{bei}_n\,\xi\right) - \mathrm{ber}_{n-1}\,\xi$$

$$\mathrm{bei}_{n+1}\,\xi = -\frac{n\sqrt{2}}{\xi}\left(\mathrm{ber}_n\,\xi + \mathrm{bei}_n\,\xi\right) - \mathrm{bei}_{n-1}\,\xi$$

$$\mathrm{ber}_n'\,\xi = -\frac{1}{\sqrt{2}}\left(\mathrm{ber}_{n-1}\,\xi + \mathrm{bei}_{n-1}\,\xi\right) - \frac{n\,\mathrm{ber}_n\,\xi}{\xi}$$

$$\mathrm{bei}_n'\,\xi = \frac{1}{\sqrt{2}}\left(\mathrm{ber}_{n-1}\,\xi - \mathrm{bei}_{n-1}\,\xi\right) - \frac{n\,\mathrm{bei}_n\,\xi}{\xi} \qquad \text{(B. 14. 16)}$$

Using these recurrent formulas, we find that the Bessel-Kelvin functions of order two are related to those of order zero by the following expressions:

$$\mathrm{ber}_2\,\xi = \frac{2}{\xi}\mathrm{bei}'\xi - \mathrm{ber}\,\xi$$

$$\mathrm{bei}_2\,\xi = -\frac{2}{\xi}\mathrm{ber}'\xi - \mathrm{bei}\,\xi$$

$$\begin{aligned}\mathrm{ber}_2'\,\xi &= -\mathrm{ber}'\xi - \frac{2\,\mathrm{ber}_2\,\xi}{\xi}\\ &= -\mathrm{ber}'\xi - \frac{4}{\xi^2}\mathrm{bei}'\xi + \frac{2}{\xi}\mathrm{ber}\,\xi\end{aligned}$$

$$\begin{aligned}\mathrm{bei}_2'\,\xi &= -\mathrm{bei}'\xi - \frac{2\,\mathrm{bei}_2\,\xi}{\xi}\\ &= -\mathrm{bei}'\xi + \frac{4}{\xi^2}\mathrm{ber}'\xi + \frac{2}{\xi}\mathrm{bei}\,\xi\end{aligned} \qquad \text{(B. 14. 17)}$$

15. Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Ellipsoidal and Conical Shape

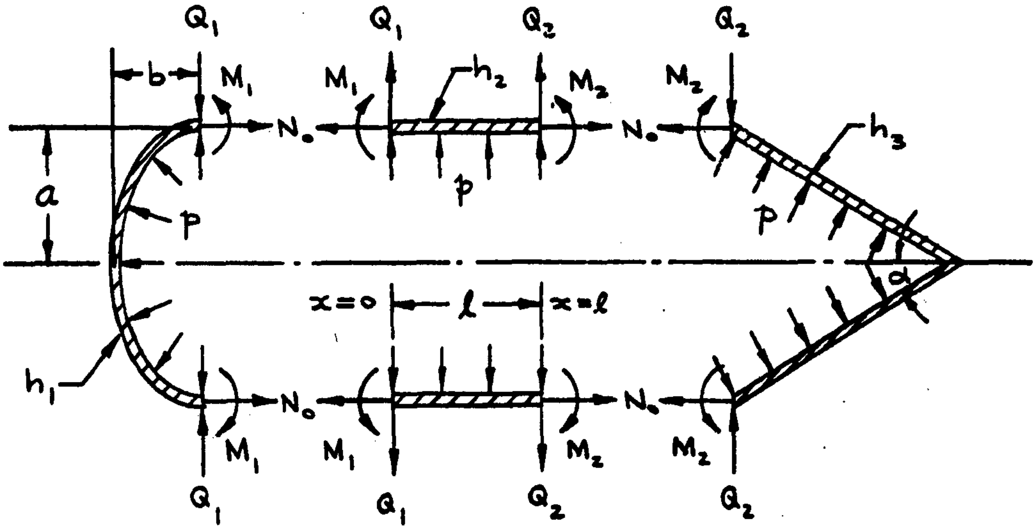


Fig. B. 15. 1 - Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Ellipsoidal and Conical Shape under Internal Pressure

The signs convention of Figs. B. 8. 2 and B. 14. 2 will be adhered to in this analysis. For the ellipsoidal head, the dimensionless forms of the radial displacement and rotation have been derived earlier, and are:

$$-\frac{Eh_1\delta_{x=0}}{4pa^2} = \frac{M_1}{4pa^2}A_{11} + \frac{Q_1}{2pa}A_{12} + A_{13} \tag{B. 15. 1}$$

$$-\frac{Eh_1^2\delta'_{x=0}}{8\beta_c^2a^3h_2p} = \frac{M_1}{4pa^2}A_{17} + \frac{Q_1}{2pa}A_{18} \tag{B. 15. 2}$$

where

$$A_{11} = -\frac{2a}{h_1}\sqrt{3(1-\nu^2)}$$

$$A_{12} = \frac{1}{\beta_{10}h_1}\sqrt{3(1-\nu^2)}$$

$$A_{13} = -\frac{1}{8}\left(2-\frac{a^2}{b^2}-\nu\right)$$

$$A_{17} = \frac{2\beta_{10}}{\beta_c^2 h_2}\sqrt{3(1-\nu^2)}$$

$$A_{18} = -\frac{\sqrt{3(1-\nu^2)}}{2\beta_c^2 a h_2}$$

$$\beta_c^4 = \frac{Eh_2}{4a^2D_c} = \frac{3(1-\nu^2)}{a^2h_2^2}$$

$$\beta_{10}^4 = \frac{Eh_1}{4a^2D_1} = \frac{3(1-\nu^2)}{a^2h_1^2}$$

$$D_c = \frac{Eh_2^3}{12(1-\nu^2)}$$

$$D_1 = \frac{Eh_1^3}{12(1-\nu^2)} \qquad \text{(B. 15. 3)}$$

For the short circular cylindrical shell, the dimensionless forms of the radial displacements and rotations at $x = 0$ and $x = \ell$ due to the forces p, Q_1, Q_2, M_1, and M_2 are, from Eqs. (B. 6. 11) through (B. 6. 14):

$$\frac{Eh_1\delta_{x=0}}{4pa^2} = A_1 + \frac{M_1}{4pa^2}(A_2\Omega_{20}) + \frac{M_2}{4pa^2}(-2A_2\Omega_{21}) + \frac{Q_1}{2pa}(A_3\Omega_{16}) + \frac{Q_2}{2pa}(A_3\Omega_{18}) \qquad \text{(B. 15. 4)}$$

$$-\frac{Eh_1^2\delta'_{x=0}}{8\beta_c^2a^3h_2p} = \frac{M_1}{4pa^2}(A_7\Omega_{17}) + \frac{M_2}{4pa^2}(-A_7\Omega_{19}) + \frac{Q_1}{2pa}(A_8\Omega_{20}) + \frac{Q_2}{2pa}(-2A_8\Omega_{21}) \tag{B. 15. 5}$$

$$\frac{Eh_3\delta_{x=\ell}}{4pa^2} = A_4 + \frac{M_1}{4pa^2}(-2A_5\Omega_{21}) + \frac{M_2}{4pa^2}(A_5\Omega_{20}) + \frac{Q_1}{2pa}(A_6\Omega_{18}) + \frac{Q_2}{2pa}(A_6\Omega_{16}) \tag{B. 15. 6}$$

$$-\frac{Eh_3^2\delta'_{x=\ell}}{8\beta_c^2a^3h_2p} = \frac{M_1}{4pa^2}(-A_9\Omega_{19}) + \frac{M_2}{4pa^2}(A_9\Omega_{17}) + \frac{Q_1}{2pa}(-2A_{10}\Omega_{21}) + \frac{Q_2}{2pa}(A_{10}\Omega_{20}) \tag{B. 15. 7}$$

where

$$A_1 = -\frac{h_1}{h_2}\left(\frac{2-\nu}{8}\right)$$

$$A_2 = -2\frac{h_1}{h_2}\beta_c^2a^2$$

$$A_3 = -\frac{h_1}{h_2}\beta_c a$$

$$A_4 = -\frac{h_3}{h_2}\left(\frac{2-\nu}{8}\right)$$

$$A_5 = -2\frac{h_3}{h_2}\beta_c^2a^2$$

$$A_6 = -\frac{h_3}{h_2}\beta_c a$$

$$A_7 = -2\beta_c a\left(\frac{h_1}{h_2}\right)^2$$

(cont.)

$$A_8 = -\frac{1}{2}\left(\frac{h_1}{h_2}\right)^2$$

$$A_9 = -2\beta_c a\left(\frac{h_3}{h_2}\right)^2$$

$$A_{10} = -\frac{1}{2}\left(\frac{h_3}{h_2}\right)^2$$

$$\Omega_{16} = \frac{\sinh\beta\ell \cosh\beta\ell - \sin\beta\ell \cos\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{17} = \frac{\sin\beta\ell \cos\beta\ell + \sinh\beta\ell \cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{18} = \frac{\cos\beta\ell \sinh\beta\ell - \sin\beta\ell \cosh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{19} = \frac{\sin\beta\ell \cosh\beta\ell + \cos\beta\ell \sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{20} = \frac{\sinh^2\beta\ell + \sin^2\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell}$$

$$\Omega_{21} = \frac{\sin\beta\ell \sinh\beta\ell}{\sinh^2\beta\ell - \sin^2\beta\ell} \qquad \text{(B. 15. 8)}$$

Finally, for the conical head, the dimensionless forms of the radial displacement and rotation at the base of the cone due to the forces p, Q_2, and M_2 are represented by:

$$-\frac{Eh_3\delta_{x=\ell}}{4pa^2} = -\frac{M_2}{4pa^2}a_1 - \frac{Q_2}{2pa}a_2 - a_3 \qquad \text{(B. 15. 9)}$$

$$-\frac{Eh_3^2\,\delta_{x=\ell}}{8pa^3\beta_e^2 h_2} = -\frac{M_2}{4pa^2}b_1 - \frac{Q_2}{2pa}b_2 - b_3 \qquad \text{(B. 15. 10)}$$

where the influence numbers a_1, a_2, a_3, b_1, b_2 and b_3 are defined in Eq. (B. 14.9).

If we now equate Eqs. (B. 15. 1) with (B. 15.4); (B. 15.2) with (B. 15.5); (B. 15.9) with (B. 15.6); and (B. 15.10) with (B. 15.7), and collecting like terms, we will obtain the following system of equations from which the desirable unknowns Q_1, Q_2, M_1, and M_2 can be determined. Indeed,

$$\frac{M_1}{4pa^2}(A_2\Omega_{20}-A_{11}) + \frac{M_2}{4pa^2}(-2A_2\Omega_{21}) + \frac{Q_1}{2pa}(A_3\Omega_{16}-A_{12}) + \frac{Q_2}{2pa}(A_3\Omega_{18}) = A_{13}-A_1$$

$$\frac{M_1}{4pa^2}(A_7\Omega_{17}-A_{17}) + \frac{M_2}{4pa^2}(-A_7\Omega_{19}) + \frac{Q_1}{2pa}(A_8\Omega_{20}-A_{18}) + \frac{Q_2}{2pa}(-2A_8\Omega_{21}) = 0$$

$$\frac{M_1}{4pa^2}(-2A_5\Omega_{21}) + \frac{M_2}{4pa^2}(A_5\Omega_{20}+a_1) + \frac{Q_1}{2pa}(A_6\Omega_{18}) + \frac{Q_2}{2pa}(A_6\Omega_{16}+a_2) = -a_3-A_4$$

$$\frac{M_1}{4pa^2}(-A_9\Omega_{19}) + \frac{M_2}{4pa^2}(A_9\Omega_{17}+b_1) + \frac{Q_1}{2pa}(-2A_{10}\Omega_{21}) + \frac{Q_2}{2pa}(A_{10}\Omega_{20}+b_2) = -b_3$$

Defining now the matrices:

$$|U| = \begin{Bmatrix} M_1 \\ M_2 \\ O_1 \\ O_2 \end{Bmatrix} \quad ; \quad |V| = \begin{Bmatrix} A_{13} - A_1 \\ o \\ -a_3 - A_4 \\ -b_3 \end{Bmatrix} \quad ; \qquad \text{(B. 15.11)}$$

and

$$[W] = \begin{bmatrix} \dfrac{A_2\Omega_{20} - A_{11}}{4pa^2} & -\dfrac{A_2\Omega_{21}}{2pa^2} & \dfrac{A_3\Omega_{16} - A_{12}}{2pa} & \dfrac{A_3\Omega_{18}}{2pa} \\[2ex] \dfrac{A_7\Omega_{17} - A_{17}}{4pa^2} & -\dfrac{A_7\Omega_{19}}{4pa^2} & \dfrac{A_8\Omega_{20} - A_{18}}{2pa} & -\dfrac{A_8\Omega_{21}}{pa} \\[2ex] -\dfrac{A_5\Omega_{21}}{2pa^2} & \dfrac{A_5\Omega_{20} + a_1}{4pa^2} & \dfrac{A_6\Omega_{18}}{2pa} & \dfrac{A_6\Omega_{16} + a_2}{2pa} \\[2ex] -\dfrac{A_9\Omega_{19}}{4pa^2} & \dfrac{A_9\Omega_{17} + b_1}{4pa^2} & -\dfrac{A_{10}\Omega_{21}}{pa} & \dfrac{A_{10}\Omega_{20} + b_2}{2pa} \end{bmatrix}$$

(B. 15. 12)

The four simultaneous equations listed above may be written in the form

$$[W]\,\{U\} = \{V\}$$

from which

$$\{U\} = [W]^{-1}\,\{V\} \tag{B.15.13}$$

16. Short Circular Cylindrical Shell with Unequal Thickness Head Closures of Hemispherical and Conical Shape

If the ellipsoidal head of Fig. B. 15. 1 is replaced by a hemispherica head of radius a and thickness h_1, the discontinuity forces Q_1, Q_2, M_1 and M_2 associated with this new configuration can be readily calculated by the formulas developed for Case 15. In this case, it is merely necessary to replace the quantity A_{13} defined in Eq. (B. 15. 3) by

$$A_{13} = -\frac{1}{8}(1 - \nu) \qquad \text{(B. 16. 1)}$$

DISTRIBUTION

Internal

J. Ancell
J. Aseltine
N. N. Au
E. J. Barlow
H. S. Braham
J. M. Brown
G. Fox
M. Goldman
A. B. Greenberg
G. E. Hlavka
V. H. Ho
B. Hohmann
J. Hook
H. Lange
E. LaPorte
E. Lenk
B. P. Leonard
E. Letsch
W. Leverton
J. Logan
A. Mager
H. Maier
J. Marsh
D. McColl
C. W. Miller
R. Morgan
B. Moss
A. G. Norem
W. Pritchard
W. Radcliffe
W. F. Roberts
T. Shiokari
W. F. Sampson
C. W. Sarture
J. Selna
M. S. Sliwinshi
H. Smallen
J. Statsinger
J. Steinman
S. Strong
W. Targoff
S. M. Tennant
C. H. Tsao
W. Tydon
V. Wall
H. K. Weiss
W. J. Wellens
W. Wetmore
R. Whalen
J. Wilder
D. Willens
J. R. Wilson

External

B/Gen J. S. Bleymaier
Col. R. C. Dineen
Lt. Col. A. J. Gardner
1st Lt. K. W. Graham
Maj. W. E. Haynes
Lt. Col. R. B. Hennessy
Maj. J. J. King
Lt. Col. A. W. Thompson

National Bureau of Standards
Boulder Labs
Boulder, Colorado

Defense Documentation Center
Cameron Station
Attn: TISIA
Alexandria, Virginia, 22314

Material Laboratory
Wright-Patterson AFB, Ohio

Scientific and Tech. Inf. Facility
Attn: NASA Rep. (SAK/DL-814)
P.O. Box 5700
Bethesda, Maryland, 20014